LECTURE 12

Agenda :

1. Binomial random variable
2. Geometric random variable

BINOMIAL RANDOM VARIABLE

Recall that a Bernoulli trial or a Bernoulli experiment has only 2 outcomes. The associated Bernoulli random variable X takes the value 0 if the outcome is a success, and 1 if the outcome is a failure. Let $P(X=1) = p$ and $P(X=0) = 1-p$, i.e., p is the probability of success for the experiment.

$E(X) = p$ and $V(X) = p(1-p)$.

Suppose an experiment consists of n independent Bernoulli trials. For eg., toss a coin 1000 times ($n = 1000$) or inspect 1000 items for being defective ($n = 1000$).

Let X = # of successes in the n trials.

If $Y_i = \begin{cases} 0 & \text{if } i^{th} \text{ trial is success,} \\ 1 & \text{if } i^{th} \text{ trial is failure.} \end{cases}$

$$X = \sum_{i=1}^{n} Y_i .$$

DEFINITION: A random variable X is said to be a BINOMIAL RANDOM VARIABLE with parameters n and p if

(i) X is the number of successes in n independent Bernoulli trials
(ii) The probability of success in each trial is p.

Let us calculate the probability mass function of a Binomial random variable.

Note that $\mathcal{X} = \{0, 1, 2, ---, n\}$.

$P(X = x) = P($Exactly x trials out of n trials result in a success$)$

$$= \underbrace{\binom{n}{x}}_{\text{\# of ways of choosing } x \text{ positions or trials}} \times \underbrace{p^x}_{\text{Probability of success in the chosen } x \text{ trials}} \times \underbrace{(1-p)^{n-x}}_{\text{Probability of failure in the remaining } n-x \text{ trials}}$$

Hence, $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, ---, n$.

IDENTITY: $\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = 1$.

Proof: $1 = \sum_{x=0}^{n} p_X(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

Recall that

$$X = \sum_{i=1}^{n} Y_i ,$$

where Y_i is the outcome of the i^{th} trial.

- $E(X) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} p = np.$

- However, from the basic definition of $E(X)$,

$$E(X) = \sum_{x \in \mathcal{X}} x p_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$\boxed{\text{IDENTITY : } \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = np}$$

FACT : If $Y_1, Y_2, ---, Y_n$ are random variables arising out of independent experiments, then

$$V\left(\sum_{i=1}^{n} Y_i \right) = \sum_{i=1}^{n} V(Y_i).$$

Hence,

$$V(X) = \sum_{i=1}^{n} V(Y_i) = \sum_{i=1}^{n} p(1-p) = np(1-p).$$

Example: A hospital has to install n generators, with the requirement that with probability 0.99 atleast one generator should be working at any given time. Suppose that the probability that any generator is operating correctly is 0.95. What is the minimum value of n?

Let X denote the number of generators which are operating correctly. Note that the probability that each generator operates correctly is 0.95. Assuming that each generator operates independently,

X is a Binomial random variable with parameters n and 0.95.

REQUIREMENT: $P(X \geq 1) = 0.99$

$$\Rightarrow \quad 1 - P(X=0) = 0.99$$

$$\Rightarrow \quad 1 - (1-p)^n = 0.99$$

$$\Rightarrow \quad (0.05)^n = 0.01$$

$$\Rightarrow \quad n = \frac{\log 0.01}{\log 0.05}$$

GEOMETRIC RANDOM VARIABLE

Consider an experiment which consists of repeating independent Bernoulli trials until a success is obtained. Assume that the probability of success in each independent trial is p.

Let X = # of failures before the first success.

$$\mathcal{X} = \text{Range}(X) = \{0, 1, 2, \text{-----------}\}$$

$$P(X = x) = P(\text{First } x \text{ trials are failure, } (x+1)^{th} \text{ trial is success})$$

$$= p(1-p)^x$$

Hence,

$$p_X(x) = p(1-p)^x \quad x = 0, 1, 2, \text{--------}$$

Note that

$$P(X = x) = p(1-p)^x = P(X = x-1)(1-p).$$

Hence, $p_X(x)$ is a decreasing function of x.

$$\boxed{\text{IDENTITY: } \quad \sum_{x=0}^{\infty} p(1-p)^x = 1.}$$

$$1 = \sum_{x=0}^{\infty} P(X = x) = \sum_{x=0}^{\infty} p(1-p)^x.$$

Let us calculate the distribution function of X

$$F_X(x) = P(X \leq x).$$

If x is a positive integer,

$$P(X \leq x) = \sum_{t=0}^{x} p(1-p)^x$$

$$= p \sum_{t=0}^{x} (1-p)^x$$

$$= \frac{p(1-(1-p)^{x+1})}{1-(1-p)}$$

$$= 1 - (1-p)^{x+1}$$

Hence for any positive integer x,

$$F_X(x) = 1 - (1-p)^{x+1}.$$