LECTURE - (14)

Agenda: 1) Negative Binomial Random Variable
2) Poisson Random Variable

EXPRESSING THE NEGATIVE BINOMIAL RANDOM VARIABLE AS A SUM OF GEOMETRIC RANDOM VARIABLES

Let W_1 denote the number of failures prior to the 1^{st} success.

Let W_2 denote the number of failures between the 1^{st} and 2^{nd} success

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Let W_r denote the number of failures between the $(r-1)^{th}$ and r^{th} success.

Since X is the total number of failures before the r^{th} success,

$$X = \sum_{i=1}^{r} W_i$$

Note that,

① W_i is a geometric random variable for $i = 1, 2, ---, r$

② $W_1, W_2, ---, W_r$ arise from independent sub-experiments.

Hence,

$$E(X) = \sum_{i=1}^{r} E(W_i) = r\frac{(1-p)}{p}.$$

$$V(X) = \sum_{i=1}^{r} V(W_i) = \frac{r(1-p)}{p^2}.$$

Example: A large lot of tires contains 5% defectives. Four tires are to be chosen from the lot and placed on a car.

(a) Find the probability that 2 defectives are found before four good ones

(b) Find the expected value and the variance of the number of defective tires chosen before finding 4 good tires.

(Assume that the number of tires is large enough so that choosing tires successively can be treated as independent and identical Bernoulli experiments).

As given to us, choosing each tire can be thought of as a Bernoulli experiment with a success if the tire is good (with probability 95%). Hence,

X = # of bad tires chosen before finding 4 good tires is a negative binomial random variable with $r = 4$, $p = 0.95$.

$$P(X=2) = \binom{4+2-1}{2}(0.95)^4(0.05)^2$$

$$= 10(0.95)^4(0.05)^2 = 0.02036$$

$$E(X) = \frac{4 \times 0.05}{0.95} = \frac{4}{19}$$

$$V(X) = 4 \times \frac{0.05}{(0.95)^2} = \frac{400}{19 \times 95} = \frac{80}{361}$$

POISSON RANDOM VARIABLE

The poisson distribution was historically derived as a limit of the binomial distribution when the number of trials $n \to \infty$, the probability of success $p \to 0$, but $np \to \lambda > 0$.

It is a very useful model for rare events. For example, the number of accidents on a highway intersection in 1 month, the number of repairs a high quality machine requires within a year, the number of cavalrymen killed by a freak accident in a year, etc.

Let us consider a binomial random variable X with parameters n and p.

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, ---, n-1, n$.

Let us assume x to be fixed. Let $n \to \infty$, $p \to 0$ and $np \to \lambda$.

$$\lim_{n\to\infty, p\to 0} P(X = x) = \lim_{n\to\infty, p\to 0} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty, p\to 0} \frac{\lambda^x}{x!} \frac{n!}{(n-x)!} \left(1 - \frac{\lambda}{n}\right)^{n-x} \frac{1}{n^x}$$

Note that, $\lim_{n\to\infty} \frac{n(n-1) --- (n-x+1)}{n^x} = 1$

Hence $\lim_{n\to\infty, p\to\infty} \frac{n!}{(n-x)!\, n^x} = 1$.

Also, $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$.

Hence,

$$\lim_{n\to\infty, p\to\infty} P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}.$$

Hence, the Poisson random variable is a random variable which takes values in $\{0, 1, 2, \ldots\ldots\}$ and

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}.$$

(The earlier calculation is just meant to be an intuitive explanation of why a Poisson random variable can be thought of as a limit of Binomial random variables under appropriate assumptions.)

$$E(X) = \sum_{x=0}^{\infty} x P(X=x) = \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda. \quad \text{(Why?)}$$

$$V(X) = \sum_{x=0}^{\infty} (x-\lambda)^2 \frac{e^{-\lambda}\lambda^x}{x!} = \lambda. \quad \text{(Why?)}$$

$$SD(X) = \sqrt{V(X)} = \sqrt{\lambda}.$$