LECTURE 15

Agenda:

1. Poisson distribution
2. Hypergeometric distribution

POISSON DISTRIBUTION

A random variable X is said to be a Poisson(λ) random variable if

(i) $\mathcal{X} = \text{Range}(X) = \{0, 1, 2, \ldots\}$

(ii) $P(X = x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$ for $x = 0, 1, 2, \ldots$

We saw in the last lecture that the poisson probabilities can be obtained as limits of binomial probabilities, when $n \to \infty$, $p \to 0$, $np \to \lambda$.

This distribution is generally used to model the number of times an event occurs in a given time frame. For example, the number of accidents at a highway intersection in one month, the number of calls passing through a cellular relay in a five minute period.

VERIFY THAT $P(X=x)$, $x = 0, 1, 2, \ldots\ldots\ldots$ add up to 1.

$$\sum_{x \in \mathcal{X}} P(X=x) = \sum_{x=0}^{\infty} P(X=x)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1.$$

Hence the probability mass function is valid.

$$E(X) = \sum_{x \in \mathcal{X}} x P(X=x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \lambda \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!}.$$

$$= \lambda \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \lambda.$$

Similarly,

$$V(X) = \lambda.$$

** and the number of repairs Y that machine B requires is Poisson random variable with mean 1.12.

<u>Example:</u> The manger of an industrial plant is planning to buy a new machine of either type A or type B. For each day's operation, the number of repairs X that machine A requires is a Poisson random variable with mean 0.96 ** The daily cost of operating A is $C_A = 160 + 40X^2$; for B, the daily cost of operating is $C_B = 128 + 40Y^2$. Assume that the repairs take negligible time and that each night the machines are cleaned so that they operate like new machines at the start of each day. Which machine minimizes the expected daily cost for the following times of daily operation?

The expected cost for machine A is

$$E[C_A(t)] = 160 + 40\,E(X^2)$$

$$= 160 + 40\left(V(X) + (E(X))^2\right)$$

$$= 160 + 40\left(0.96 + (0.96)^2\right)$$

$$= 235.264.$$

The expected cost for machine B is

$$E[C_B(t)] = 128 + 40\,E(Y^2)$$

$$= 128 + 40\left(1.12 + (1.12)^2\right)$$

$$= 222.976$$

Example: The number of calls coming into a hotel's reservation center is a Poisson random variable with mean 3. Find the probability that no calls arrive in a given 1-minute period.

X = # of calls in the given one-minute period

Given that X is Poisson(3).

$$P(X = 0) = \frac{e^{-3}\,3^{0}}{0!} = e^{-3}.$$

Assuming that the number of calls in each minute behaves independently, find the probability that atleast two calls will arrive in a given two-minute period.

Let X_1 = # of calls in the first minute
X_2 = # of calls in the second minute

We are required to find $P(X_1 + X_2 \geq 2)$

$$P(X_1 + X_2 \geq 2) = 1 - P(X_1 + X_2 < 2)$$

$$= 1 - P(X_1 + X_2 = 0) - P(X_1 + X_2 = 1)$$

$$= 1 - P(X_1 = 0, X_2 = 0) - P(X_1 = 0, X_2 = 1) - P(X_1 = 1, X_2 = 0)$$

$$= 1 - e^{-3}e^{-3} - e^{-3}\frac{e^{-3}\,3}{1!} - \frac{e^{-3}\,3}{1!}e^{-3}$$

$$= 1 - e^{-6} - \frac{e^{-6}\,6}{1!}$$

$$= 1 - 7e^{-6}.$$

$$= 0.983$$

HYPERGEOMETRIC DISTRIBUTION

All distributions that we have discussed are in some or the other way related to the Bernoulli distribution. For these random variables, we consider repeated Bernoulli trials which for all practical purposes, are independent and identical. Here is a situation when this is not the case.

Suppose we have a small lot consisting of N items, of which k are defective. Suppose that n items are sampled randomly and sequentially from the lot WITHOUT REPLACEMENT. Let X denote the number of defective items in the n items that are chosen.

$$P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\underbrace{\binom{N}{n}}_{\text{Total number of outcomes}}} \longrightarrow \text{\# of outcomes with } x \text{ defectives}$$