LECTURE 18

Agenda:

(1) Properties of probability density functions
(2) Examples

PROPERTIES OF PROBABILITY DENSITY FUNCTIONS

We saw that continuous random variables cannot be described in terms of probability mass functions, as this leads to inconsistencies with the three basic axioms of probability. As a solution, the concept of a probability density function for continuous random variables was introduced.

Definition: If X is a continuous random variable, then there exists a function f_X such that

(i) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ ($f_X(x) = 0$ for $x \notin \mathcal{X} = \text{Range}(X)$)

(ii) $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

(iii) $$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx \text{ for any } a < b \in \mathbb{R}.$$

We start by looking at some properties of the probability density function.

We saw that if $F_X(b) = P(X \leq b)$ is the probability distribution function of a continuous random variable, then

$$f_X(b) = \frac{d}{db} F_X(b).$$

Hence,

$$f_X(b) = \lim_{h \to 0} \frac{F_X(b+h) - F_X(b-h)}{2h}$$ (by definition of derivatives)

$$= \lim_{h \to 0} \frac{P(X \leq b+h) - P(X \leq b-h)}{2h}$$

$$= \lim_{h \to 0} \frac{P(b-h < X \leq b+h)}{2h}$$

Hence, for any two points b and b',

$$\frac{f_X(b)}{f_X(b')} = \underbrace{\lim_{h \to 0} \frac{P(b-h < X \leq b+h)}{P(b'-h < X \leq b'+h)}}$$

$$\boxed{\text{Ratio of the probability the } X \text{ is very close to } b\text{, to the probability that } X \text{ is very close to } b'.}$$

~~Hence, although the probability density function does not directly relate to~~

Hence, the ratio of the probability density function at two points can be roughly interpreted as the ratio of the probabilities that X is close to the two points respectively.

If X is a continuous random variable with density f_X, then

$$P(X = x) = P(x \leq X \leq x) = \int_x^x f_X(y)\,dy = 0.$$

Hence, we inherently assign $P(X = x) = 0$ for every x.

CONFUSED ? Remember that we are in a difficult situation and are trying to come up with a reasonable framework for describing continuous random variables probabilistically. The framework we have suggested is indeed the best possible way out.

1. Note that X can take uncountably many values. Hence saying $P(X=x) = 0$ DOES NOT RULE OUT x as a possible value (ATLEAST THAT IS NOT HOW WE SHOULD INTERPRET IT)

2. Note that as long as $x \in \mathcal{X} = \text{Range}(X)$,

$$P(x-h \leq X \leq x+h) = \int_{x-h}^{x+h} f_X(y)\,dy > 0.$$

Hence, for continuous random variables, any interval (in the range of X) is assigned a positive probability, but every point in the range of X is assigned zero probability.

EXAMPLES

Example 1: The distribution function of the random variable X, the time (in months) from the diagnosis age until death for a population of patients with AIDS, is as follows

$$F_X(x) = \begin{cases} 1 - e^{-0.03x^{1.2}}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(a) Find the probability density function of X

Note that for $x \geq 0$,

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\left(1 - e^{-0.03x^{1.2}}\right)$$

$$= 0.03 \times 1.2 x^{0.2} e^{-0.03x^{1.2}}$$

$$= 0.036 x^{0.2} e^{-0.03x^{1.2}},$$

and for $x < 0$,

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}(0) = 0.$$

Hence, the density of X is given by

$$f_X(x) = \begin{cases} 0.036 x^{0.2} e^{-0.03x^{1.2}} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(b) Find the probability that a randomly selected person survives atleast 12 months.

$$P(X \geq 12) = 1 - P(X \leq 12) \quad (\because P(X=12)=0)$$

$$= 1 - F_X(12)$$

$$= 1 - \left(1 - e^{-0.03(12)^{1.2}}\right)$$

$$= 0.55$$

Example 2: Suppose that a random variable X has a probability density function given by

$$f_X(x) = \begin{cases} \frac{x^2}{3} & \text{if } -1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the probability that $-1 < X < 1$.

$$P(-1 < X < 1) = P(-1 \leq X \leq 1) \quad (\because P(X=-1) = P(X=1) = 0)$$

$$= \int_{-1}^{1} f_X(x)\,dx$$

$$= \int_{-1}^{1} \frac{x^2}{3}\,dx$$

$$= \left[\frac{x^3}{9}\right]_{-1}^{1}$$

$$= \frac{1-(-1)}{9}$$

$$= \frac{2}{9}.$$

[b] Find the distribution function of X.

Note that for $x \leq -1$,

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(y)\,dy = 0.$$

$$(\because f_X(y) = 0 \text{ if } y < -1)$$

For $-1 < x < 2$,

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

$$= \int_{-1}^{x} \frac{y^2}{3}\,dy$$

$$= \left[\frac{y^3}{9}\right]_{-1}^{x}$$

$$= \frac{x^3+1}{9}$$

For $x \geq 2$,

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

$$= \int_{-1}^{2} f_X(y)\,dy$$

$$= \frac{2^3+1}{9}$$

$$= 1.$$

Hence,

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ \frac{x^3+1}{9} & \text{if } -1 < x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

<u>Example 3:</u> Suppose that the weekly repair cost for a certain machine (in units of \$100) denoted by the random variable X, has probability density function given by

$$f_X(x) = \begin{cases} cx(1-x), & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find c.

Since f_X is a probability density function,

$$\int_{-\infty}^{\infty} f_X(x) = 1.$$

Hence, $$\int_0^1 cx(1-x)\,dx = 1$$

$$\Rightarrow \quad c\left\{ \int_0^1 x\,dx - \int_0^1 x^2\,dx \right\} = 1$$

$$\Rightarrow \quad c\left\{ \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 \right\} = 1$$

$$\Rightarrow \quad c\left\{ \frac{1}{2} - \frac{1}{3} \right\} = 1$$

$$\Rightarrow \quad c = 6.$$