# LECTURE 20

**Agenda:**

1. The uniform distribution
2. The exponential distribution

In the previous few lectures, we have been studying continuous random variables and various identities related to them. We will start studying specific continuous random variables which are useful in practice.

## UNIFORM DISTRIBUTION

The simplest model which corresponds to a continuous random variable is the uniform distribution. Suppose we pick a point randomly from an interval $(a, b)$, i.e., all points are "equally likely" to be chosen. Let $X$ = Point chosen in $(a, b)$.

How do we convert the informal understanding that all points are "equally likely" to a precise statement about the distribution function and density function of $X$?

Note that since all points are "equally likely" to be chosen, the probability that $X$ lies in a given interval should only depend on the length of the interval. Hence, for $a \leq c < d \leq b$,

$$\mathcal{X} = \text{Range}(X) = (a, b)$$

$$P(X \in (c,d)) = \frac{d-c \;\to \text{Length of } (c,d)}{b-a \;\to \text{Length of } (a,b)}$$

Hence, for $x < a$,

$$F_X(x) = P(X \leq x) = 0$$

For $a \leq x \leq b$

$$F_X(x) = P(X \leq x) = P(a \leq X \leq x) = \frac{x-a}{b-a}$$

For $b < x$

$$F_X(x) = P(X \leq x) = P(a \leq X \leq b) = \frac{b-a}{b-a} = 1.$$

The distribution function of a Uniform $(a, b)$ random variable is given by

$$F_X(x) = \begin{cases} 0 & \text{if } x < a, \\ \dfrac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } b < x. \end{cases}$$

By differentiating the distribution function, we obtain that the probability density function of a Uniform$(a,b)$ random variable is given by

$$\boxed{f_X(x) = \begin{cases} 0 & \text{if} \quad x < a \text{ or } x > b, \\ \frac{1}{b-a} & \text{if} \quad a \leq x \leq b. \end{cases}}$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{a+b}{2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^3 - a^3}{2(b-a)} - \left(\frac{a+b}{2}\right)^2$$

Hence, $V(X) = \dfrac{(b-a)^2}{12}$.

<u>Example:</u> A bomb is to be dropped along a 1-mile long line that streches across a practice target zone. The target zone's center is at the midpoint of the line. The target will be destroyed if the ~~bomb falls at a random location along the line.~~ bomb falls within 0.1 mile on either side of the center. Find the probability that the target will be destroyed given that the bomb falls at a random location along the line.

Let a and b denote the endpoints of the target zone. Then $b - a = 1$.

Let $X$ = Location of the bomb.

$$P(\text{Target is destroyed}) = P\left(X \in \left(\frac{a+b}{2} - 0.1, \frac{a+b}{2} + 0.1\right)\right)$$

$$= \frac{\left(\frac{a+b}{2} + 0.1\right) - \left(\frac{a+b}{2} - 0.1\right)}{b-a}$$

$$= \frac{0.2}{1}$$

$$= 0.2.$$

# EXPONENTIAL DISTRIBUTION

In many practical applications, especially while studying the lifetimes of ~~[illegible]~~ certain objects, we encounter random variables whose probability of lying in intervals of constant length decreases exponentially as the interval is moved to the right. These random quantities are modeled by the exponential distribution.

A random variable $X$ is called an Exponential$(\theta)$ random variable, if for some fixed $\theta > 0$,

$$\mathcal{X} = \text{Range}(X) = (0, \infty).$$

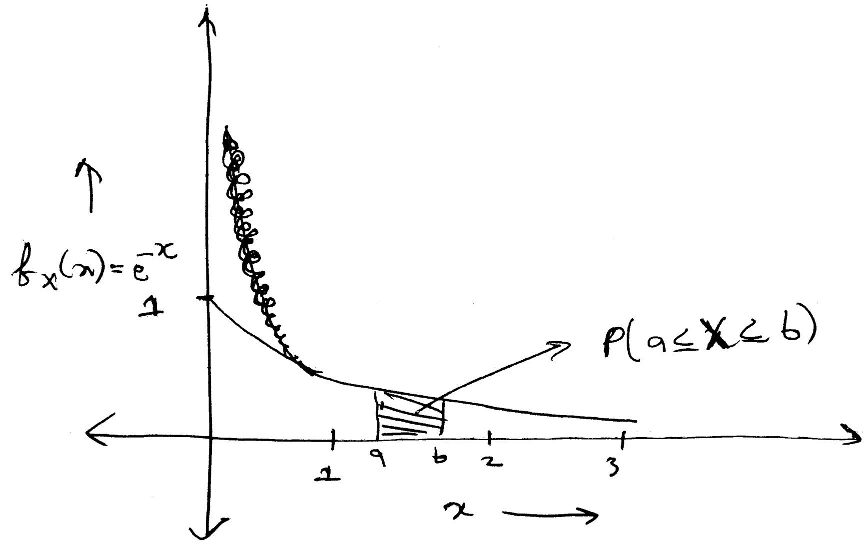
$$F_X(x) = \begin{cases} 1 - e^{-x/\theta} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The probability density function of an ~~[illegible]~~ exponential random variable is given by

$$f_X(x) = \begin{cases} \dfrac{e^{-x/\theta}}{\theta} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The parameter $\theta$ is generally called the scale parameter for the exponential distribution.

PLOT OF THE DENSITY FUNCTION FOR AN EXPONENTIAL (1) RANDOM VARIABLE




The plot of the density function for a continuous random variable often gives us a good idea of how the probability structure of the random variable is defined. Note that

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

= The area under the plot of $f_X$ from $a$ to $b$.