LECTURE - 21

Agenda:

1. Exponential distribution
2. Gamma distribution

EXPONENTIAL DISTRIBUTION

Let us recollect that the exponential random variable is used generally to model lifetimes of objects. [redacted] An exponential random variable with parameter $\theta > 0$ has the following properties

1. $\mathcal{X} = \text{Range}(X) = [0, \infty)$.
2. $F_X(x) = \begin{cases} 1 - e^{-x/\theta} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$
3. $f_X(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$
4. $E(X) = \theta$.

The derivation of $E(X)$ was done by using Gamma functions.

Definition: The function $\Gamma : (0, \infty) \to (0, \infty)$ is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x} \, dx$.

FACT: $\Gamma(n) = (n-1)!$ for every positive integer n.

Hence,
$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = 0! = 1$$
$$\Gamma(2) = \int_0^\infty x e^{-x}\,dx = 1! = 1$$
$$\Gamma(3) = \int_0^\infty x^2 e^{-x}\,dx = 2! = 2$$
$$\Gamma(4) = \int_0^\infty x^3 e^{-x}\,dx = 3! = 6$$

and so on.

$$V(X) = \int_0^\infty \frac{x^2}{\theta} e^{-x/\theta}\,dx - (E(X))^2$$

Substituting $y = \frac{x}{\theta}$, we get

$$V(X) = \theta^2 \int_0^\infty y^2 e^{-y}\,dy - \theta^2$$
$$= \theta^2\,\Gamma(3) - \theta^2$$
$$= 2\theta^2 - \theta^2$$
$$= \theta^2.$$

MEMORYLESS PROPERTY OF THE EXPONENTIAL DISTRIBUTION

Note that we had seen the memoryless property of the geometric distribution. It is the only discrete distribution with this property. The exponential distribution also shares this property. It is the only continuous distribution with this property. To see this,

$$P(X \geq a+b \mid X \geq a) = \frac{P(X \geq a+b)}{P(X \geq a)}$$

$$= \frac{1-(1-e^{-\frac{a+b}{\theta}})}{1-(1-e^{-\frac{a}{\theta}})}$$

$$= e^{-\frac{b}{\theta}}$$

$$= P(X \geq b).$$

Hence, given that an exponential random variable is greater than a, the conditional probability that it is greater than $a+b$ is the same as the unconditional probability that it is greater than b.

Example: The magnitudes of earthquakes recorded in a region of North America can be modeled by an exponential distribution with a mean of 2.4, as measured on the Richter scale.

(a) Find the probability that the next earthquake will be no more than 2.5 on the Richter scale.

Note that if X = Magnitude of next earthquake, then X is an exponential random variable with parameter 2.4. Hence,

$$P(X \leq 2.5) = 1 - e^{-\frac{2.5}{2.4}} = \quad .$$

(b) Given that the next earthquake will be more than 2 on the Richter scale, what is the probability that it will be more than 3 on the Richter scale.

By the memoryless property of the exponential distribution,

$$\begin{aligned} P(X \geq 3 \mid X \geq 2) &= P(X \geq 1) \\ &= e^{-\frac{1}{2.4}} \\ &= \quad . \end{aligned}$$

THE GAMMA DISTRIBUTION

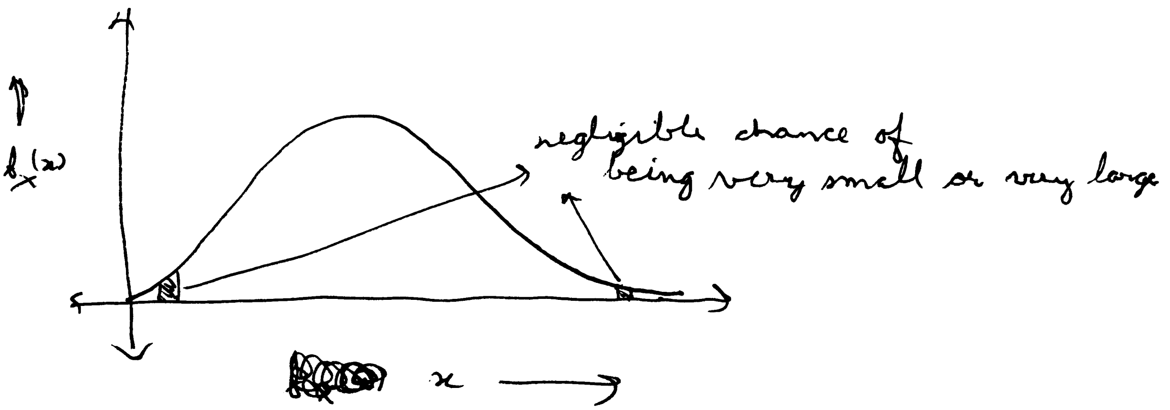
For many random variables, the probability of being around zero is very small. The probability increases as we move away from zero, and then decreases again as we get close to large values. In other words, the chance of observing very small or very large values is negligible, and most of the values observed are close to the average associated with the random variable

A random variable X is said to be a Gamma random variable if

$$\mathcal{X} = \text{Range}(X) = (0, \infty)$$

$$f_X(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Here $\alpha, \beta > 0$ are fixed parameters for the distribution



Note that when $\alpha = 1$, the gamma density reduces

to the exponential density with parameter β.
Hence the exponential random variable is a special case of the gamma random variable with $\alpha = 1$.

The first thing that should be verified is whether the density integrates to 1.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$$

(Substitute $y = \frac{x}{\beta}$)

$$= \int_0^{\infty} \frac{\cancel{\beta^\alpha}}{\Gamma(\alpha)\cancel{\beta^\alpha}} y^{\alpha-1} e^{-y}\, dy$$

$$= \int_0^{\infty} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)}\, dy$$

$$= 1.$$