LECTURE - 22

Agenda:

1. Gamma distrubution
2. Normal distribution

GAMMA DISTRIBUTION

The material presented below was covered in the previous lecture. Let us recollect that X is called a gamma random variable with parameters α and β if

$$\mathcal{X} = \text{Range}(X) = (0, \infty),$$

$$f_X(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

$$\boxed{\text{FACT: If } \alpha > 1, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$$

We use this fact to derive $E(X)$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_0^\infty x \cdot \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^\alpha e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^\alpha e^{-y}\, dy$$

$\left(\because \text{Substitute } y = \frac{x}{\beta}\right)$

$$= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1)$$

$$= \beta\alpha.$$

$\left(\because \Gamma(\alpha+1) = \alpha\Gamma(\alpha)\right)$

One can also establish that $V(X) = \beta^2\alpha$.

NORMAL DISTRIBUTION

The most widely used continuous random variable in probability and its applications is the "normal random variable". A random variable X is said to be a normal random variable with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, if

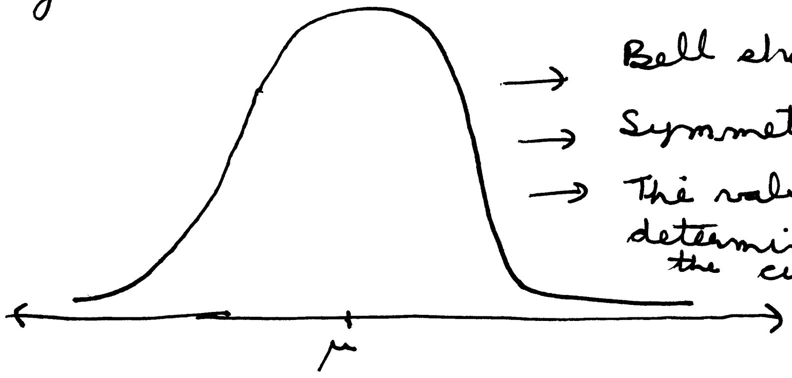
$\mathcal{X} = \text{Range}(X) = \mathbb{R}$, and

$$f_X(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \qquad x \in \mathbb{R}.$$

What do the parameters μ and σ mean in terms of the random variable X?

$$\boxed{\text{RESULT:} \quad E(X) = \mu \ \text{ and } \ V(X) = \sigma^2}$$

Let us postpone the proof of this result for later and look at the shape of the normal density.



→ Bell shaped curve

→ Symmetric around μ

→ The value of σ determines the spread of the curve

The normal distribution works as a good model for a lot of measurements that are observed in real experiments. There is a valid reason for this and we will see that a few weeks later, but for now let us contend ourselves with the knowledge that if the random quantity under consideration is an average of independent random quantities, then the normal distribution is quite likely an appropriate model for that random quantity.

THE STANDARD NORMAL DISTRIBUTION

<u>Definition</u>: If Z is a normal random variable with parameters $\mu = 0$ and $\sigma = 1$, then Z is said to be a "standard normal random variable".

Note that, by definition, the density of a standard normal random variable is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } z \in \mathbb{R}.$$

Let us calculate $E[Z]$ and $V(Z)$.

$$E[Z] = \int_{-\infty}^{\infty} z \, f_Z(z)\, dz$$

$$= \int_{-\infty}^{\infty} \frac{z e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

Here is a nice trick to evaluate this integral. Substitute $y = -z$.

$$E[Z] = -\int_{-\infty}^{\infty} \frac{y e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy = -E[Z].$$

Hence, $E[Z] = 0$.

The main reason which this trick worked is that

$g(z) = \dfrac{z e^{-z^2/2}}{\sqrt{2\pi}}$ is an "odd function", i.e.,

$$g(z) = \frac{z e^{-z^2/2}}{\sqrt{2\pi}} = -\left\{ \frac{(-z)\, e^{-(-z)^2/2}}{\sqrt{2\pi}} \right\} = -g(-z) \text{ for } z \in \mathbb{R}.$$

$$
\begin{aligned}
V(Z) &= E(Z^2) - (E(Z))^2 \\
&= E(Z^2) \\
&= \int_{-\infty}^{\infty} z^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} u^{1/2} e^{-\frac{u}{2}}\, du
\end{aligned}
$$

($\because$ Substitute $u = z^2$)

$$= \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{3}{2}\right) (2)^{3/2}$$

($\because$ By the definition of the gamma function)

$$= 1$$

($\because$ By the property of the gamma function)

RESULT: If Z is a standard normal random variable, then for any $\mu \in \mathbb{R}$ and $\sigma > 0$,

$X = \mu + \sigma Z$ is a normal random variable with parameters μ and σ, i.e., the probability density function of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in \mathbb{R}.$$

Proof of $E(X) = \mu$ and $V(X) = \sigma^2$

Remember the result that we had stated earlier about $E(X)$ and $V(X)$. Let us prove the result using the property described above.

$$E(X) = E(\mu + \sigma Z) = \mu + \sigma E(Z) = \mu + 0 = \mu$$

$$V(X) = V(\mu + \sigma Z) = V(\sigma Z) = \sigma^2 V(Z) = \sigma^2$$