LECTURE - 24

<u>Agenda</u>

1. The standard normal distribution function and its properties
2. Examples

THE STANDARD NORMAL DISTRIBUTION FUNCTION

Let us recollect that Z is said to be a standard normal random variable, if the probability density of Z is given by

$$f_Z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \text{for } z \in \mathbb{R}.$$

The distribution function of a standard normal random variable has a special name and place in the theory of probability.

Definition: The " Φ - function" from $\mathbb{R}$ to $[0,1]$ is defined as

$$\Phi(z) = \int_{-\infty}^{z} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = P(Z \le z)$$

for $z \in \mathbb{R}$.

Hence Φ is essentially the distribution function of a standard normal random variable. It cannot be evaluated in closed form (except at special values), but with modern computing technology there are programs which will give you the value of Φ at any given point. Let us study some of the important properties of this function.

PROPERTY 1: $0 \leq \Phi(z) \leq 1$ for $z \in \mathbb{R}$

This follows as $\Phi(z) = P(Z \leq z)$ where Z is $N(0,1)$

notation for standard normal

PROPERTY 2: $\Phi(z) + \Phi(-z) = 1$

$$\Phi(z) + \Phi(-z) = \int_{-\infty}^{z} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx + \int_{-\infty}^{-z} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$$

$$= \int_{-\infty}^{z} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy + \int_{\infty}^{z} \frac{e^{-\frac{(-y)^2}{2}}}{\sqrt{2\pi}}\,d(-y)$$

(Substitute $y = x$ in the first integral) (Substitute $y = -x$ in the second integral)

$$= \int_{-\infty}^{z} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy + \int_{z}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$= 1.$$

PROPERTY 3: If X is a normal random variable with mean μ and variance σ^2, then

$$F_X(x) = P(X \le x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Note that if Z is $N(0,1)$, then $\mu + \sigma Z$ is $N(\mu, \sigma^2)$. Hence,

$$P(X \le x) = P(\mu + \sigma Z \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Example: Sick leave time used by employees of a firm in the course of 1 month has approximately a normal distribution with a mean of 180 hours and a variance of 350 hours. Find the probability that the total sick leave for a given month is less than 150 hours. Your answer may be expressed in terms of the Φ-function.

Let X = Total sick leave time.

Then X is $N(150, 350)$. $(\mu = 150, \sigma^2 = 350)$.

$$P(X < 150) = P(X \le 150) \quad (\because X \text{ is continuous r.v. hence } P(X=150)=0)$$

$$= \Phi\left(\frac{150-180}{\sqrt{350}}\right)$$

$$= \Phi\left(-\frac{30}{\sqrt{350}}\right).$$

$$= 0.05440$$

Note that $\Phi(0) + \Phi(-0) = 1$. Hence $\Phi(0) = P(Z \le 0) = \frac{1}{2}$.

Hence, if we want to find the probability that X is less than its mean 180, then

$$P(X < 180) = P(X \le 180)$$

$(\because X$ is a continuous random variable, hence $P(X = 180) = 0)$

$$= \Phi\left(\frac{180-180}{\sqrt{350}}\right)$$

$$= \Phi(0)$$

$$= \frac{1}{2}$$

Here are some other facts about the Φ-function.

$$\Phi(1) - \Phi(-1) = P(-1 \le Z \le 1) \approx 68\%.$$

$$\Phi(2) - \Phi(-2) = P(-2 \le Z \le 2) \approx 95\%.$$

$$\Phi(3) - \Phi(-3) = P(-3 \le Z \le 3) = 99.7\%.$$

This means that if X is $N(\mu, \sigma^2)$, then

$$P(\mu - \sigma \le X \le \mu + \sigma) = P\left(-1 \le \frac{X-\mu}{\sigma} \le 1\right)$$

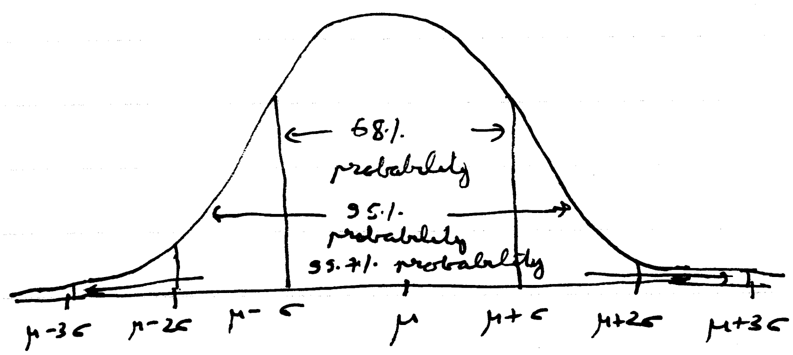
$$= P(-1 \le Z \le 1)$$

$$\approx 68\%$$

Similarly,

$$P(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 95\%.$$

$$P(\mu \mp 3\sigma \le X \le \mu + 3\sigma) = 99.7\%.$$



Hence if X is $N(\mu, \sigma^2)$, there is 99.7% probability that it takes a value in the interval $[\mu - 3\sigma, \mu + 3\sigma]$. Since the length of the interval $[\mu - 3\sigma, \mu + 3\sigma]$ is 6σ, it is used as the "6σ" rule often used in quality control divisions.

__Example:__ Suppose that men's neck sizes are approximately normally distributed with a mean of 16.2 inches and variance of 0.81 inches. Find the probability that the neck size of a randomly chosen man lies between 13.5 inches and 18.9 inches.

X = Man's neck size.

X is $N(16.2,\ 0.81)$. $(\mu = 16.2,\ \sigma^2 = 0.81)$

$$P(13.5 \le X \le 18.9) = P\left(-3 \le \frac{X - 16.2}{0.9} \le 3\right)$$

$$= P(-3 \le Z \le 3)$$

$$\approx 99.7\%$$