LECTURE 25

Agenda

1. Moment generating functions (MGFs)
2. Properties

In the last lecture, we defined the notion of a moment generating functions. Today, we will derive the moment generating functions for some standard random variables, and learn a very useful property of moment generating functions

BINOMIAL MGF

If X is Binomial (n, p), then

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= (pe^t + 1-p)^n$$

(By the binomial theorem,
$$\sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} = (a+b)^n \quad)$$

Note that

$$\frac{d}{dt} M_X(t) = \frac{d}{dt}\left[(pe^t + 1-p)^n\right]$$
$$= npe^t (pe^t + 1-p)^{n-1}$$

Hence,

$$\left.\frac{d}{dt} M_X(t)\right|_{t=0} = npe^0(pe^0 + 1-p)^{n-1}$$
$$= np.$$
$$= E[X]$$

We just verified that the first derivative of the moment generating function evaluated at 0, gives us the expected value

STANDARD NORMAL MGF

If Z IS NORMAL$(0,1)$, then

$$M_Z(t) = E[e^{tZ}]$$

$$= \int_{-\infty}^{\infty} e^{tz} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

$$= e^{\frac{t^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\frac{(t-z)^2}{2}}}{\sqrt{2\pi}} dz}_{\text{Density of Normal}(\mu=t,\ \sigma^2=1)}$$

$$= e^{\frac{t^2}{2}}.$$

GAMMA MGF

If X IS GAMMA(α, β), then

$$M_X(t) = E[e^{tX}]$$

$$= \int_0^{\infty} e^{tx} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} dx$$

$$= \int_0^{\infty} \frac{x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)}}{\Gamma(\alpha)\beta^{\alpha}} dx$$

Note that if $t \geq \frac{1}{\beta}$, then the integral is infinite. If $t < \frac{1}{\beta}$, then

$$M_X(t) = \int_0^\infty \frac{x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)}}{\Gamma(\alpha)\beta^\alpha} dx$$

Let $y = x\left(\frac{1}{\beta} - t\right)$ Then,

$$M_X(t) = \frac{\left(\frac{1}{\beta}-t\right)^{-\alpha}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{\alpha-1} e^{-y} dy$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\alpha)(1-\beta t)^\alpha}$$

$$= \frac{1}{(1-\beta t)^\alpha}.$$

PROPERTY OF MGF: If X and Y are two random variables, such that

$$M_X(t) = M_Y(t) \text{ for every } t \in \mathbb{R},$$

(assuming that $M_X(t) < \infty$ for an interval around 0), then X and Y have the same distribution.

APPLICATION:

Let Z be STANDARD NORMAL. We are interested in finding the distribution of Z^2. Note that

$$M_{Z^2}(t) = E\left[e^{tZ^2}\right]$$

$$= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}(1-2t)}\, dz$$

This is infinity if $t \geq \frac{1}{2}$.

If $t < \frac{1}{2}$

$$= \frac{1}{\sqrt{1-2t}} \left\{ \int_{-\infty}^{\infty} \underbrace{\sqrt{\frac{1-2t}{2\pi}}\, e^{-\frac{z^2}{2}(1-2t)}}_{\text{Density of Normal}\left(\mu=0,\ \sigma^2=\frac{1}{1-2t}\right)} dz \right\}$$

$$= \frac{1}{\sqrt{1-2t}}$$

This is exactly the MGF of a Gamma$\left(\alpha=\frac{1}{2}, \beta=2\right)$ random variable.

Hence Z^2 has a Gamma$\left(\alpha=\frac{1}{2}, \beta=2\right)$ distribution.