LECTURE - 26

Agenda:

1. Moment generating functions
2. Mixed random variables

MOMENT GENERATING FUNCTIONS

Let us recollect that a moment generating function $M_X : \mathbb{R} \to (0, \infty)$ for a random variable X is defined by

$$M_X(t) = E[e^{tX}] = \begin{cases} \sum_{x \in \mathcal{X}} e^{tx} P(X = x) & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx & \text{if } X \text{ is continuous.} \end{cases}$$

It is named so, because the moments of the random variable can be obtained by differentiating it an appropriate number of times, i.e.,

$$\left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0} = E(X^k) \quad \text{for every positive integer } k.$$

We derived in class that if X is a Uniform random variable on the interval $[a, b]$, then

$$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

Note that

$$\frac{d}{dt} M_X(t) = \frac{b e^{tb} - a e^{ta}}{t(b-a)} - \frac{e^{tb} - e^{ta}}{t^2(b-a)}$$

$$= \frac{1}{t^2(b-a)} \left\{ (bt-1) e^{tb} - (at-1) e^{ta} \right\}$$

$$= \frac{1}{t^2(b-a)} \left\{ (bt-1) \sum_{x=0}^{\infty} \frac{(tb)^x}{x!} - (at-1) \sum_{x=0}^{\infty} \frac{(ta)^x}{x!} \right\}$$

$$\left(\because e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \right)$$

$$= \frac{1}{t^2(b-a)} \left\{ \frac{(b^2-a^2)t^2}{2} + \text{terms with } t^3 \text{ or higher power of } t \right\}$$

$$= \frac{a+b}{2} + \frac{1}{(b-a)} \left\{ \text{terms with } t \text{ or higher power of } t \right\}$$

Hence, $\left.\frac{d}{dt} M_X(t)\right|_{t=0} = \frac{a+b}{2} = E(X).$

Moment generating function of a standard normal random variable

Let Z be $N(0,1)$. Then

$$M_Z(t) = E\left[e^{tZ}\right] = \int_{-\infty}^{\infty} e^{tz} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{t^2}{2}} e^{tz} e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(z-t)^2}{2}}}{\sqrt{2\pi}}\, dz$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy$$

($\because$ Substitute $y = z - t$)

$$= e^{\frac{t^2}{2}}.$$

(The density $\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$ integrates to 1.)

If X is $N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma}$ has the same distribution as a standard normal random variable. This will simplify the computation of the moment generating function of X greatly.

$$M_X(t) = E[e^{tX}]$$

$$= e^{\mu t} E[e^{t(X-\mu)}]$$

$$= e^{\mu t} E[e^{\sigma t \underbrace{\frac{(X-\mu)}{\sigma}}_{\text{Standard normal}}}]$$

$$= e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

($\because$ By using the formula for the moment generating function of a standard normal random variable)

Let us calculate the moment generating function for a discrete random variable.

Let X be a Poisson random variable with mean λ, i.e., $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, 2, \ldots$

Then,

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} P(X=x)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} e^{tx} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}.$$

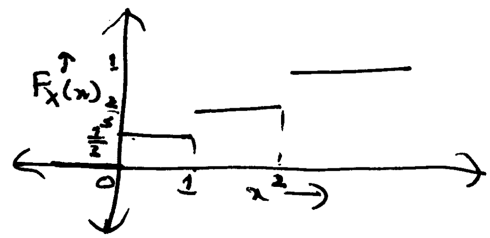
The moment generating functions are really useful in studying theoretical properties of random variables. For example, one can easily prove using moment generating functions, that if Z is $N(0,1)$, then Z^2 has the same distribution as a Gamma random variable with $\alpha = \frac{1}{2}$ and $\theta = 2$.

Mixed random variables

Until now, we have studied two kinds of random variables, discrete and continuous. Here is a characterization of discrete and continuous random variables in terms of their distribution functions.

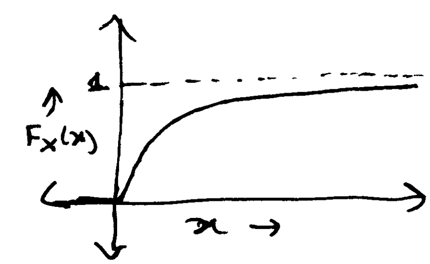
<u>Result:</u> A random variable X is discrete if and only if its distribution function is a piecewise constant function with positive jumps at points in $\mathcal{X}$ = Range (X).

An example:
$P(X=0) = P(X=1) = P(X=2) = \frac{1}{3}$



<u>Result:</u> A random variable X is continuous if and only if its distribution function is a continuous function (piecewise differentiable).

An example:
Exponential distribution



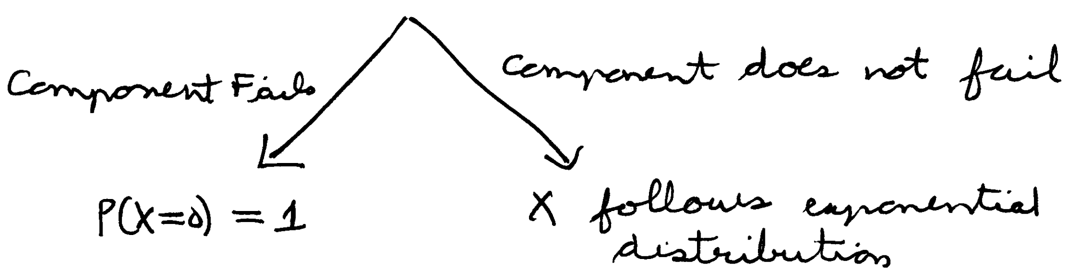
However, there are many random variables, whose distribution functions do not behave in either of the two ways written above. Such random variables are called mixed random variables.

Here is a simple example of a mixed random variable.

Let X denote the life-length (in hundreds of hours) of a certain type of electronic component. These components frequently fail immediately upon insertion into the system. The probability of immediate failure is $\frac{1}{4}$. However, if a

component does not fail immediately, its life-length distribution has the exponential density.

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$



Hence, for any real number x, by Bayes rule

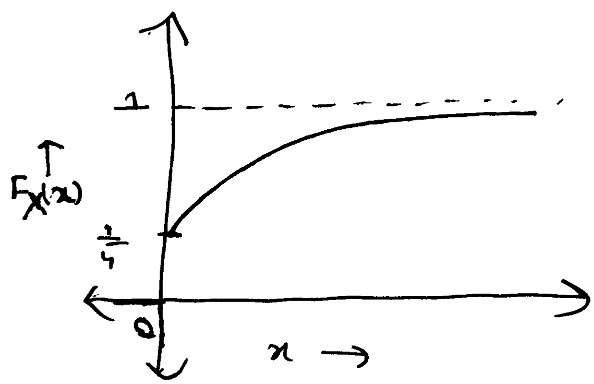
$$\begin{aligned} P(X \leq x) &= P(X \leq x \mid C.F.)\, P(C.F.) \\ &\quad + P(X \leq x \mid C.D.N.F)\, P(C.D.N.F) \\ &= \frac{1}{4} F_1(x) + \frac{3}{4} F_2(x), \end{aligned}$$

where $$F_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

Probability distribution function of a discrete random variable with $P(X=0) = 1$

and $$F_2(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 - e^{-x} & \text{if } x > 0. \end{cases}$$

Hence $$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{4} + \frac{3}{4}(1 - e^{-x}) & \text{if } x \geq 0. \end{cases}$$



Does not satisfy the properties of the distribution function of a discrete or continuous random variable.