

LECTURE - 33

Agenda:

- ① The multinomial distribution
- ② Moment generating functions and sums of independent random variables.

THE MULTINOMIAL DISTRIBUTION

Until now we have developed methods to study the joint probability behaviour of two random variables. But often we have experiments where we are interested in the joint probability behaviour of not just two, but several random variables. Let us study one example of such an experiment.

- ① Suppose the experiment consists of n independent trials
- ② Suppose each trial has k possible outcomes, and the probability of outcome i is p_i , for every $i=1, 2, \dots$

(If $k=2$, then we have a binomial experiment.)

Let $X_i = \#$ of outcomes which are i , for every $i=1, 2, \dots$

The random variables X_1, X_2, \dots, X_k completely describe the randomness in the experiment, and hence it is very natural to ~~consider~~ study the joint probability behaviour of these random variables. As in the case of two random variables, the joint probability behaviour of the discrete random variables X_1, X_2, \dots, X_k is described by their joint probability mass function

$$p_{X_1, X_2, \dots, X_k}(x_1, x_2, \dots, x_k) \triangleq P(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k)$$

for all $x_i \geq 0$ such that $\sum_{i=1}^k x_i = n$.

Note that,

$$P(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k)$$

$$= P(x_1 \text{ outcomes which are 1, } x_2 \text{ outcomes which are 2, } \dots, x_k \text{ outcomes which are } k)$$

$$= \frac{n!}{x_1! x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

of ways of partitioning n outcomes into k groups of size x_1, x_2, \dots, x_k respectively.

The vector (X_1, X_2, \dots, X_k) is said to follow the multinomial distribution with parameters n and p_1, p_2, \dots, p_k .

RESULT: If (X_1, X_2, \dots, X_k) follows a multinomial distribution with parameters n and p_1, p_2, \dots, p_k , then

$$E(X_i) = np_i, \quad V(X_i) = np_i(1-p_i), \text{ and}$$

$$\text{Cov}(X_i, X_j) = -np_i p_j, \text{ for } 1 \leq i \neq j \leq k.$$

Example: The National Fire Incident Reporting Service says that among residential fires, approximately 74% are in 1 or 2 family homes, 20% are in multifamily homes, and 6% in other dwellings. If five fires are reported independently in a day, find the probability that 2 are in 1 or 2 family homes, 2 are in multifamily homes and one is in other dwellings.

Clearly, this is a multinomial experiment.

Let $X_1 = \#$ of 1 or 2 family home fires
 $X_2 = \#$ of multifamily home fires
 $X_3 = \#$ of other fires.

Then, (X_1, X_2, X_3) is multinomial with parameters 5 and 0.74, 0.20, 0.06.

Often, moment generating functions are useful for finding the distribution of sums of independent random variables. Here are two examples.

Example: Let X_1 and X_2 be independent exponential random variables, with mean θ . Find the distribution of $Y = X_1 + X_2$.

Let us find the moment generating function of Y .

$$\begin{aligned} M_Y(t) &= E[e^{tY}] \\ &= E[e^{t(X_1+X_2)}] \\ &= E[e^{tX_1} e^{tX_2}] \\ &= E[e^{tX_1}] E[e^{tX_2}] \\ &\quad (\because \text{By independence of } X_1 \text{ and } X_2) \end{aligned}$$

$$= \frac{1}{1-\theta t} \cdot \frac{1}{1-\theta t}$$

(\because By the formula for the moment generating function of an exponential random variable)

$$= \frac{1}{(1-\theta t)^2}$$

But note that moment generating functions characterize the distribution of a random variable, and by

observing that $\frac{1}{(1-t)^2}$ is the moment generating

function of a Gamma random variable with parameters $\alpha = 2$ and $\beta = \theta$, we conclude that

Y follows a Gamma distribution with parameters $\alpha = 2$ and $\beta = \theta$.

Example 2: Let X_1 and X_2 be independent Poisson random variables. ^{\rightarrow Let X_1 have mean λ_1 and X_2 have mean λ_2 .} ~~Find the distribution of $Y = X_1 + X_2$.~~ Find the distribution of $Y = X_1 + X_2$.

Let us ~~get~~ get the moment generating function of Y .

$$M_Y(t) = E[e^{tY}]$$

$$= E[e^{t(X_1 + X_2)}]$$

$$= E[e^{tX_1} e^{tX_2}]$$

$$= E[e^{tX_1}] E[e^{tX_2}]$$

(\because By independence of X_1 and X_2)

$$= e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^t - 1)}$$

(~~∴~~ ∴ By the formula for the moment generating function of a Poisson random variable;

$$= e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

Again, note that moment generating functions characterize the distribution of a random variable.

Also, $e^{\lambda(e^t - 1)}$ is the moment generating function of a Poisson random variable with mean λ .

~~∴~~ This clearly implies Y follows a Poisson distribution with mean $\lambda_1 + \lambda_2$.

