LECTURE - 34

Agenda:

1. Functions of discrete random variables
2. Functions of continuous random variables (Method of distribution functions)

FUNCTIONS OF DISCRETE RANDOM VARIABLES

We have seen that in many experiments, we have one ~~[illegible]~~ or more ~~[illegible]~~ random variables of interest and we often wish to study their joint behaviour. Along the same lines, we are often interested in the probability distribution of a function of these random variables. Let us consider the discrete random variable case first.

Let X be a discrete random variable with p.m.f. $p_X(x) = P(X = x)$ for every $x \in \mathcal{X} = \text{Range}(X)$. Suppose we are interested in the probability distribution of $Y = f(X)$.

General algorithm:

1. Identify the range of Y.
2. For every $y \in \mathcal{Y} = \text{Range}(Y)$, find $p_Y(y) = P(Y = y)$

by expressing the event $\{Y = y\}$ in terms of the random variable X.

<u>**Example:**</u> A quality control manager samples from a large lot of items, testing each item until r defectives have been found. Find the distribution of Y, the number of items that are tested to obtain r defectives.

Assuming that the lot is large, this experiment is a negative binomial experiment and if

X = # of non-defectives found before r defectives are found,

then

$$P(X = x) = \binom{x+r-1}{r-1} p^r q^x, \quad x = 0, 1, 2, ----$$

where p is the probability of finding a defective item.

But we are interested in Y which is the total number of items that are tested to obtain r defectives.

Note that $Y = X + r$.

$$\mathcal{Y} = \text{Range}(Y) = \{r, r+1, r+2, ---------\}.$$

For every $y \in \{r, r+1, r+2, \dots\dots\dots\}$

$$P(Y = y) = P(X + r = y)$$

$$= P(X = y - r)$$

$$= \binom{y - r + r - 1}{r - 1} p^r q^{y-r}$$

$$= \binom{y - 1}{r - 1} p^r q^{y-r}.$$

FUNCTIONS OF CONTINUOUS RANDOM VARIABLES (METHOD OF DISTRIBUTION FUNCTIONS)

We have to more careful when dealing with continuous random variables, as they are expressed in terms of ~~[illegible]~~ density functions.

Let X be a continuous random variable with p.d.f f_X. Suppose we are interested in the probability density function of $Y = f(X)$.

General algorithm:

(1) Express the event $\{Y \leq y\}$ in terms of X.

(2) Using this express ~~[illegible]~~ $F_Y(y) = P(Y \leq y)$ as

an expression in terms of F_X.

③ Differentiate to get $f_Y(y) = \frac{d}{dy} F_Y(y)$.

<u>Example:</u> Suppose X is a continuous random variable with probability density f_X, and we want to find the probability density function for $Y = X^2$.

① For every $y \geq 0$,

$$\begin{aligned} P(Y \leq y) &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

For every $y < 0$,

$$P(Y \leq y) = 0.$$

② Hence,

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

③ Differentiating with respect to y, we get that,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y}) & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right) & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

<u>Example:</u> The proportion of time X that a coffee machine is in use during a typical 40-hour workweek is a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} 3x^2, & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The actual number of hours out of a 40-hour week that the coffee machine is not in use is given by

$$Y = 40(1 - X).$$

Find the probability density function of Y.

(1) $\{Y \leq y\} = \{40(1-X) \leq y\}$
$= \left\{ X \geq 1 - \frac{y}{40} \right\}$

(2) Hence,

$$F_Y(y) = P(Y \le y)$$

$$= P\left(X \ge 1 - \frac{y}{40}\right)$$

$$= \int_{1-\frac{y}{40}}^{1} f_X(x)\,dx$$

$$= \begin{cases} 0 & y < 0, \\ 1 - \left(1 - \frac{y}{40}\right)^3 & 0 \le y \le 40, \\ 1 & y > 40. \end{cases}$$

(3) Differentiating with respect to y, we get that,

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \begin{cases} \frac{d}{dy}\left\{-\left(1 - \frac{y}{40}\right)^3\right\} & \text{if } 0 \le y \le 40, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{3}{40}\left(1 - \frac{y}{40}\right)^2 & \text{if } 0 \le y \le 40, \\ 0 & \text{otherwise.} \end{cases}$$