LECTURE - 35

Agenda:

1. Method of transformations
2. Method of conditioning

METHOD OF TRANSFORMATIONS

~~Let us recollect the method of transformations from the previous lecture.~~

Given: A continuous random variable X with density f_X given by

$$f_X(x) = \begin{cases} > 0 & \text{if } x \in (a,b), \\ 0 & \text{otherwise.} \end{cases}$$

Task: Find the probability density function of $Y = g(X)$, where g is a one-to-one differentiable function on the range of f_X.

General algorithm:

1. Identify (α, β), where (α, β) is the image of (a,b) under g.
2. Construct the inverse function h, by solving for X in $Y = g(X)$, and expressing it as $X = h(Y)$.

(3) Verify that $h'(y) = \frac{d}{dy} h(y) \neq 0$ for each $y \in (\alpha, \beta)$.

(4) Compute the probability density function of Y as

$$f_Y(y) = \begin{cases} f_X(h(y))\,|\underbrace{h'(y)}_{\frac{d}{dy}h(y)}| & \text{if } y \in (\alpha, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

<u>Example:</u> Let X be a random variable with probability density function given by

$$f_X(x) = \begin{cases} (3/2)\,x^2, & \text{if } -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the density function of $Y = 3X$.

(1) $g(x) = 3x, \quad (a, b) = (-1, 1)$.
The image of (a, b) under g is $(\alpha, \beta) = (-3, 3)$.

(2) $Y = 3X \Rightarrow X = \frac{Y}{3}$.

Hence $h(y) = \frac{y}{3}$.

③ $h'(y) = \frac{1}{3} \neq 0$ for every $y \in (-3,3)$.

④ $$f_Y(y) = \begin{cases} f_X(h(y))\,|h'(y)| & \text{if } y \in (\alpha, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f_X\left(\frac{y}{3}\right) \left|\frac{1}{3}\right| & \text{if } y \in (-3,3), \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{y^2}{18} & \text{if } y \in (-3,3), \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find the probability density function of $Y = 3 - X$

① $g(x) = 3 - x$, $(a,b) = (-1,1)$.
The image of $(-1,1)$ under g is $(\alpha, \beta) = (2,4)$.

② $Y = 3 - X \Rightarrow X = 3 - Y$.

Hence, $h(y) = 3 - y$.

③ $h'(y) = -1 \neq 0$ for every $y \in (2,4)$

④ $$f_Y(y) = \begin{cases} f_X(h(y))\,|h'(y)| & \text{if } y \in (\alpha, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f_X(3-y)\,|-1| & \text{if } y \in (2,4), \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3}{2}(3-y)^2 & \text{if } y \in (2,4), \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the probability density function of $Y = X^3$.

(1) $g(x) = x^3$, $(a, b) = (-1, 1)$.
The image of $(-1, 1)$ under g is $(\alpha, \beta) = (-1, 1)$.

(2) $Y = X^3 \Rightarrow X = \sqrt[3]{Y}$.

Hence $h(y) = \sqrt[3]{y}$.

(3) $h'(y) = \frac{1}{3} \frac{1}{(\sqrt[3]{y})^2} \neq 0$ for every $y \in (-1, 1)$.

(4) $$f_Y(y) = \begin{cases} f_X(h(y))\, |h'(y)| & \text{if } y \in (\alpha, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f_X(\sqrt[3]{y}) \left| \frac{1}{3(\sqrt[3]{y})^2} \right| & \text{if } y \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{2} & \text{if } y \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

METHOD OF CONDITIONING

Conditional probability density functions can also be used to find probability density functions of specific functions of random variables. The basis of this method is the following observation.

Suppose X_1 and X_2 are continuous random variables with joint probability density function f_{X_1,X_2}. Suppose we want to find the density function of $Y = g(X_1, X_2)$. Note that if f_{Y,X_2} is the probability density function of Y and X_2,

$$f_Y(y) = \int_{-\infty}^{\infty} f_Y(y, x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} \underbrace{f_{Y|X_2=x_2}(y)}_{} \, f_{X_2}(x_2)\, dx_2$$

Find this by the method of transformations.

General algorithm:

① Fix $X_2 = x_2$. Find the conditional probability density function of X_1 given $X_2 = x_2$.

② Note that $Y = g(X_1, x_2)$ is a transformation of X_1. Assuming it is one-to-one and differentiable,

fixed

Find the conditional probability density function of Y given $X_2 = x_2$.

③ Compute the probability density functions of Y as

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X_2=x_2}(y)\, f_{X_2}(x_2)\, dx_2 .$$

Example: Let X_1 and X_2 be independent random variables each following the exponential distribution with mean 1. Find the density function of $Y = \frac{X_1}{X_2}$.

① Fix $X_2 = x_2 > 0$. Since X_1 is independent of X_2, the conditional probability density function of X_1 given $X_2 = x_2$, is the same as the marginal probability density function of X_1, ~~given by~~

Hence, the conditional probability density function is given by

$$f_{X_1|X_2=x_2}(x_1) = \begin{cases} e^{-x_1} & \text{if } x_1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

② Given $X_2 = x_2$, $Y = \frac{X_1}{x_2}$. Let us apply the method of transformations.

① $g(x_1) = \frac{x_1}{x_2}$, $(a, b) = (0, \infty)$

The image of (a, b) under g is $(\alpha, \beta) = (0, \infty)$.

② $Y = \frac{X_1}{x_2} \Rightarrow X_1 = x_2 y$.

Hence, $h(y) = x_2 y$

③ $h'(y) = x_2 \neq 0$ for every $y \in (0, \infty)$.

④ $$f_{Y|X_2=x_2}(y) = \begin{cases} f_{X_1|X_2=x_2}(h(y))\,|h'(y)| & \text{if } y \in (\alpha, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f_{X_1|X_2=x_2}(x_2 y)\,|x_2| & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} x_2 e^{-x_2 y} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

③

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X_2=x_2}(y)\, f_{X_2}(x_2)\, dx_2$$

$$= \begin{cases} \int_0^{\infty} x_2 e^{-x_2 y} e^{-x_2}\, dx_2 & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \int_0^{\infty} x_2 e^{-x_2 (y+1)}\, dx_2 & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{1}{(y+1)^2} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$