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METHOD OF MOMENT GENERATING FUNCTIONS

· Moment generating functions can be used to identify the distribution of a sum of independent random variables. The main result that forms the basis of this approach says that if X and Y are two random variables such that their moment generating functions are the same (on an interval around zero), then X and Y have the same probability distribution function. We use this property to establish various identities that are useful in applications. We first recall a simple identity about moment generating functions.

Result: If $X_1, X_2, ---, X_n$ are independent random variables, then

$$M_{X_1+X_2+---+X_n}(t) = M_{X_1}(t) M_{X_2}(t) --- M_{X_n}(t)$$

The proof of this identity is straightforward, but will be omitted due to time constraints.

EXAMPLES

<u>Example 1:</u> Let $X_1, X_2, ---, X_n$ be independent random variables which have a gamma distribution with parameters α and β. Find the distribution of $X_1 + X_2 + --- + X_n$.

$$M_{X_1+X_2+---+X_n}(t) = E\left[e^{t(X_1+X_2+----+X_n)} \right]$$

$$= E\left[e^{tX_1} e^{tX_2} --- e^{tX_n}\right]$$

$$= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] --- E\left[e^{tX_n}\right]$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha} \left(\frac{1}{1-\beta t}\right)^{\alpha} --- \left(\frac{1}{1-\beta t}\right)^{\alpha}$$

$$= \left(\frac{1}{1-\beta t}\right)^{n\alpha}.$$

But this is precisely the moment generating function of a gamma random variable, with parameters $n\alpha$ and β. Hence, $X_1 + X_2 + --- + X_n$ has a gamma distribution with parameters $n\alpha$ and β.

Example 2: Let $X_1, X_2, \ldots, X_n$ be independent normal random variables, where $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$. Find the distribution of $X_1 + X_2 + \cdots + X_n$

Note that if $X \sim N(\mu, \sigma^2)$, then

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

Hence,

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}}\, e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} \cdots e^{\mu_n t + \frac{\sigma_n^2 t^2}{2}}$$

$$= e^{\left(\sum_{i=1}^{n} \mu_i\right) t + \frac{\left(\sum_{i=1}^{n} \sigma_i^2\right) t^2}{2}}.$$

But this is precisely the moment generating function of a normal random variable with parameters $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$. Hence, $X_1 + X_2 + \cdots + X_n$ has a normal distribution with parameters $\sum_{i=1}^{n} \mu_i$ and $\sum_{i=1}^{n} \sigma_i^2$.

Example 3: Let Z be a standard normal random variable. Find the distribution of Z^2.

$$M_{Z^2}(t) = E[e^{tZ^2}]$$

$$= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2(1-2t)}\, dz$$

$$= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \underbrace{\sqrt{\frac{1-2t}{2\pi}}\, e^{-\frac{1}{2}z^2(1-2t)}\, dz}_{\text{density of a normal random variable with parameters } \mu=0,\ \sigma^2 = \frac{1}{1-2t}}$$

(holds only if $t < \frac{1}{2}$)

$$= \frac{1}{\sqrt{1-2t}}.$$

But this is precisely the moment generating function of a gamma random variable with parameters $\alpha = \frac{1}{2}$ and $\beta = 2$.

If $Z_1, Z_2, ---, Z_n$ are independent standard normal random variables, then it follows that,

$$M_{\sum_{i=1}^{n} Z_i^2}(t) = M_{Z_1^2}(t)\, M_{Z_2^2}(t) --- M_{Z_n^2}(t),$$

$$= \frac{1}{\sqrt{1-2t}} \cdot \frac{1}{\sqrt{1-2t}} \cdots \frac{1}{\sqrt{1-2t}}$$

$$= \left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$$

But this is precisely the moment generating function of a gamma random variable with parameters $\alpha = \frac{n}{2}$ and $\beta = 2$. These random variables have a special name.

<u>Definition:</u> A gamma distribution with parameters $\alpha = \frac{n}{2}$ and $\beta = 2$ is known as a chi-square distribution with n degrees of freedom.

<u>REMARK:</u> The chi-square distribution appears frequently in applications. An example is provided below.

Let $X_1, X_2, ---, X_n$ be a set of independent random variables having a normal distribution with parameters μ and σ^2.

Suppose we observe $X_1, X_2, ---, X_n$ and want to estimate μ and σ^2.

The sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is taken to be the estimator of μ, and the sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is taken to be the estimator of σ^2. We know that $\overline{X}$ has a normal distribution with $E(\overline{X}) = \mu$ and $V(\overline{X}) = \frac{\sigma^2}{n}$.

Using the method of moment generating functions, it can be proved that $\frac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with $(n-1)$ degrees of freedom, and $\overline{X}$ and S^2 are independent random variables. This example shows the strength of the moment generating functions approach