SOLUTIONS TO SAMPLE EXAM

PROBLEM 1: (a) Let $\mu$ denote the average life of the electronic component in the population. The above problem can be formulated as testing the hypotheses

$$H_0 : \mu = 10 \quad \text{v.s.} \quad H_A : \mu > 10$$

based on a random sample $Y_1, Y_2, \ldots, Y_{100}$ of electronic components

(b) Since $\mu$ is the population average, an appropriate test statistic is the sample mean

$$\bar{Y} = \frac{1}{100} \sum_{i=1}^{100} Y_i$$

(c) Based on the large sample testing procedure derived in class, the rejection region of the level-0.05 test based on $\bar{Y}$ is given by

$$\text{Reject } H_0 \text{ if } \bar{Y} > \underbrace{10 + z_{0.95} \frac{S}{\sqrt{100}}}$$

OBSERVED VALUE $\rightarrow$ 16 $\qquad 10 + 1.68 \times \frac{7}{10} = 11.7$

Since $16 > 11.76$, it follows that there is enough evidence to reject $H_0$, and therefore accept the researcher's claim.

**Problem 2:** $\boxed{(a)}$ Note that the rejection region of the standard level-0.05 test for this problem is given by

$$\text{Reject } H_0 \text{ if } \quad \bar{Y} > 18 + 1.68 \times \frac{S}{\sqrt{36}}$$

Based on the formula provided in class,

$$\beta(20) = \Phi\left(\frac{18-20}{5/\sqrt{36}} + z_{0.95}\right)$$

$$= \Phi\left(-\frac{12}{5} + 1.68\right)$$

$$= \Phi(-2.4 + 1.68)$$

$$= \Phi(-0.72)$$

$\boxed{(b)}$ Based on the formula provided in class, the minimum sample size to ensure $\beta(20) = 0.01$ is given by

$$n = \frac{(z_{0.95} + z_{0.99})^2 \, 25}{(20-18)^2} = \frac{(1.68+2.3)^2 \times 25}{4}$$

$$= 99.025$$

Problem 3: $\boxed{a}$ We will reject the null hypothesis for large values of $\hat{\theta}$.

$\boxed{b}$ Based on the formula provided in class, the p-value is given by

$$P\left(\hat{\theta} \geq \hat{\theta}_{observed} \,\middle|\, H_0 \text{ is true}\right)$$

$$= P\left(\frac{\hat{\theta} - 10}{\hat{SE}(\hat{\theta})} \geq \frac{11-10}{\hat{SE}(\hat{\theta})} \,\middle|\, \theta = 10\right)$$

$$= P\left(\text{Normal}(0,1) \geq \frac{11-10}{5} \,\middle|\, \theta = 10\right)$$

$$= 1 - \underbrace{\Phi}_{\text{CDF of Normal}(0,1)}\left(\frac{1}{5}\right).$$

$\boxed{c}$ Since $1 - \Phi\left(\frac{1}{5}\right) > 0.01$, we will accept $H_0$ at level $0.01$.

Problem 4: $\boxed{a}$ By the Neyman-Pearson lemma, the rejection region for the most powerful level-$\alpha$ test is given by

$$\left\{(y_1, y_2, \ldots, y_n) : \frac{L(\theta_0 \mid y_1, \ldots, y_n)}{L(\theta_A \mid y_1, \ldots, y_n)} \leq k\right\}$$

Hence, we reject for small values of $\dfrac{L(\theta_0 \mid y_1, \ldots, y_n)}{L(\theta_A \mid y_1, \ldots, y_n)}$

Note that

$$\frac{L(\theta_0 \mid y_1, ---, y_n)}{L(\theta_A \mid y_1, ---, y_n)} = \frac{\prod_{i=1}^{n} f_{Y_i}(y_i \mid \theta_0)}{\prod_{i=1}^{n} f_{Y_i}(y_i \mid \theta_A)}$$

$$= \frac{\prod_{i=1}^{n} \frac{1}{\theta_0} e^{-y_i/\theta_0}}{\prod_{i=1}^{n} \frac{1}{\theta_A} e^{-y_i/\theta_A}}$$

$$= \underbrace{\left(\frac{\theta_A}{\theta_0}\right)^n e^{\sum_{i=1}^{n} y_i \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right)}}$$

INCREASING ~~DECREASING~~ FUNCTION OF $\sum_{i=1}^{n} y_i$, SINCE $\frac{1}{\theta_A} - \frac{1}{\theta_0} > 0$.

Hence, the MP level-$\alpha$ test rejects $H_0$ for small values of $\frac{L(\theta_0 \mid y_1, ---, y_n)}{L(\theta_A \mid y_1, ---, y_n)}$, or

equivalently for small values of $\sum_{i=1}^{n} y_i$.

$\boxed{\text{(b)}}$ The rejection region of the MP level-$\alpha$ test is of the form $\{\sum y_i \leq k'\}$, where $k'$ is obtained from the level $= \alpha$ constraint. Hence $k'$ DOES NOT DEPEND ON $\alpha$.