SOLUTIONS TO SAMPLE EXAM CONTINUOUS RANDOM VARIABLES

Problem 1: (a) X is Exponential with mean 9. Hence the parameter $\theta = 9$.

$$f_X(x) = \begin{cases} \frac{1}{9} e^{-\frac{x}{9}} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) **Method I:**

$$P(25 \leq X \leq 45 \mid X > 20) = P(X \geq 25 \mid X \geq 20) - P(X \geq 45 \mid X \geq 20)$$

$$\left(\because \{25 \leq X \leq 45\} \cup \{X \geq 45\} = \{X \geq 25\} \right)$$

$$= P(X \geq 5) - P(X \geq 25)$$

$$\left(\because \text{By the memoryless property } P(X \geq s+t \mid X \geq t) = P(X \geq s) \right)$$

$$= e^{-\frac{5}{9}} - e^{-\frac{25}{9}}.$$

Method II:

$$P(25 \leq X \leq 45 \mid X > 20) = \frac{P\left(\{25 \leq X \leq 45\} \cap \{X > 20\}\right)}{P(X > 20)}$$

$$= \frac{P(25 \leq X \leq 45)}{P(X > 20)}$$

$$= \frac{\int_{25}^{45} f_X(x)\, dx}{e^{-\frac{20}{9}}}$$

$$= \frac{\int_{25}^{45} \frac{1}{9} e^{-\frac{x}{9}}\, dx}{e^{-\frac{20}{9}}}$$

$$= \frac{e^{-\frac{25}{9}} - e^{-\frac{45}{9}}}{e^{-\frac{20}{9}}}$$

$$= e^{-\frac{5}{9}} - e^{-\frac{25}{9}}$$

(c) $E[X(X+3)] = E[X^2 + 3X]$

$$= E[X^2] + 3E[X]$$

$$= V(X) + (E[X])^2 + 3E[X]$$

$$= \theta^2 + \theta^2 + 3\theta$$

$$= 189.$$

Problem 2: (a) Let X = Point where car stops. Then X is uniformly distributed in the interval $[a, b]$.

(a) P(Car stops closer to a than to b)

$$= P(X - a < b - X)$$

$$= P\left(X < \frac{a+b}{2}\right)$$

$$= \int_{-\infty}^{\frac{a+b}{2}} f_X(x)\, dx$$

$$= \int_{a}^{\frac{a+b}{2}} \frac{1}{b-a}\, dx$$

$$= \frac{\frac{a+b}{2} - a}{b - a}$$

$$= \frac{1}{2}.$$

(b) P (Car stops at a point where the distance to a is more than 9 times the distance of b)

$$= P(X - a > 9(b - x))$$

$$= P\left(X > \frac{a+9b}{10}\right)$$

$$= \int_{\frac{a+9b}{10}}^{\infty} f_X(x)\,dx$$

$$= \int_{\frac{a+9b}{10}}^{b} \frac{1}{b-a}\,dx$$

$$= \frac{b - \left(\frac{a+9b}{10}\right)}{b-a}$$

$$= \frac{1}{10}.$$

(c) Let X_1, X_2, X_3 denote the stopping points of the 3 cars. Then, X_1, X_2, X_3 are independent and all of them have a uniform distribution on the interval $[a,b]$.

P(Exactly one of three travels past midpoint)

$$= P\left(X_1 > \frac{a+b}{2}, X_2 < \frac{a+b}{2}, X_3 < \frac{a+b}{2}\right) + P\left(X_1 < \frac{a+b}{2}, X_2 > \frac{a+b}{2}, X_3 < \frac{a+b}{2}\right)$$

$$+ P\left(X_1 < \frac{a+b}{2}, X_2 < \frac{a+b}{2}, X_3 > \frac{a+b}{2}\right)$$

$$= 3P\left(X_1 \geq \frac{a+b}{2}\right) P\left(X_2 < \frac{a+b}{2}\right) P\left(X_3 < \frac{a+b}{2}\right) \quad (\because \text{By independence})$$

$$= 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{3}{8}.$$

Problem 3: (a)

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$E(X) = \alpha\beta = 20$$
$$V(X) = \alpha\beta^2 = 100$$

$$\Rightarrow \alpha = 4, \quad \beta = 5.$$

(c)

$$P\left(X \in \left[E(X) - k\sqrt{V(X)},\ E(X) + k\sqrt{V(X)}\right]\right) \geq 1 - \frac{1}{k^2}$$

We want $1 - \dfrac{1}{k^2} = 0.99$.

Hence, $k = 10$.

$$P\left(X \in \left[E(x) - 10\sqrt{V(x)},\ E(x) + 10\sqrt{V(x)}\right]\right) \geq 99\%.$$

$$\Rightarrow P\left(X \in [20 - 10 \times 10,\ 20 + 10 \times 10]\right) \geq 99\%.$$
$$\Rightarrow P\left(X \in [-80,\ 120]\right) \geq 99\%.$$

Since X is a non-negative random variable,

$$P(X \in [0, 120]) \geq 99\%.$$

Problem 4: (a) Let X = Length of randomly caught fish.

$P($ Fisherman catches spotted sea trout within the legal limits$)$

$$= P(14 \leq X \leq 30)$$

IMPORTANT IDEA: ALWAYS CONVERT FROM NORMAL TO STANDARD NORMAL. NOTE THAT,

$$X = \sigma Z + \mu = 4Z + 22,$$

WHERE Z IS A STANDARD NORMAL RANDOM VARIABLE.

$$= P(14 \leq 4Z + 22 \leq 30)$$

$$= P(-2 \leq Z \leq 2)$$

$$\approx 95\%.$$

(By identity provided in class)

(b) We have to find x such that

$$P(X \leq x) = 0.5$$

or $$\Phi\left(\frac{x-\mu}{\sigma}\right) = 0.5$$

or $$\Phi\left(\frac{x-22}{4}\right) = \Phi(0)$$

Clearly $x = 22$ is the correct choice.

(c) P(Three trouts outside legal limits, fourth inside legal limits)

= P(First outside) P(Second outside) P(Third outside) P(Fourth inside) ($\because$ By independence)

$= (1-0.95)\times(1-0.95)\times(1-0.95)\times 0.95$ ($\because$ By Part (a))

$= (0.05)^3 \times (0.95)$