SOLUTIONS TO SAMPLE EXAM : JOINT DISTRIBUTIONS

Problem 1: $\boxed{(a)}$

$$f_Y(0.7) = \int_{-\infty}^{\infty} f_{X,Y}(x, 0.7)\,dx$$

Note that

$$f_{X,Y}(x, 0.7) = \begin{cases} 2 & \text{if } 0 \le x \le 0.3, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\cancel{f_Y(0.7)} = \int_0^{0.3} 2\,dx$$

$$= 0.6 \, .$$

$\boxed{(b)}$

$$f_{X|Y=0.7}(x) = \frac{f_{X,Y}(x, 0.7)}{f_Y(0.7)}$$

$$= \begin{cases} \frac{2}{0.6} & \text{if } 0 \le x \le 0.3, \\ \frac{0}{0.6} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{10}{3} & \text{if } 0 \le x \le 0.3, \\ 0 & \text{otherwise.} \end{cases}$$

$\boxed{(c)}$

$$P(X > 0.95 \mid Y = 0.7) = \int_{0.95}^{\infty} f_{X|Y=0.7}(x)\,dx$$

$$= \int_{0.95}^{\infty} 0\,dx$$

$$= 0.$$

<u>Problem 2:</u> $\boxed{(a)}$

$$P(X<2, Y>1) = \iint_A f_{X,Y}(x,y)\,dx\,dy,$$

where $A = \{(x,y) : x < 2,\ y > 1\}$.

We want to replace $f_{X,Y}(x,y)$ by e^{-x}. To do that, we need to consider the region

$$A \cap \{(x,y) : 0 \le y \le x < \infty\}$$

$$= \{(x,y) : x<2,\ y>1,\ 0 \le y \le x < \infty\}$$

$$= \underbrace{\{(x,y) : 1 < y < 2,\ y \le x \le 2\}}_{\text{HOW DID I GET THIS?}},$$

First find the range for y. Then for a fixed y, find the range for x.

Hence,

$$P(X<2, Y>1) = \int_1^2 \int_y^2 e^{-x}\,dx\,dy$$

$$= \int_1^2 (e^{-y} - e^{-2})\, dy$$

$$= (e^{-1} - e^{-2}) - e^{-2}$$

$$= e^{-1} - 2e^{-2}.$$

$\boxed{(b)}$ $$P(X \geq 2Y) = \iint_A f_{X,Y}(x,y)\, dx\, dy,$$

where $A = \{(x,y) : x \geq 2y\}$.

We want to replace $f_{X,Y}(x,y)$ by e^{-x}. To do that, we need to consider the region

$$A \cap \{(x,y) : 0 \leq y \leq x < \infty\}$$

$$= \{(x,y) : x \geq 2y,\ 0 \leq y \leq x < \infty\}$$

$$= \{(x,y) : 0 \leq y < \infty,\ 2y \leq x < \infty\}.$$

Hence,

$$P(X \geq 2Y) = \int_0^\infty \int_{2y}^\infty e^{-x}\, dx\, dy$$

$$= \int_0^\infty e^{-2y}\, dy$$

$$= \left[-\frac{1}{2} e^{-2y} \right]_0^{\infty}$$

$$= 0 - \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}.$$

$\boxed{(c)}$ $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

Note that $f_{X,Y}(x,y) = \begin{cases} e^{-x} & \text{if } 0 \le y \le x < \infty, \\ 0 & \text{otherwise.} \end{cases}$

Hence,

$$f_X(x) = \begin{cases} \int_0^x e^{-x}\, dy & \text{if } 0 \le x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} x e^{-x} & \text{if } 0 \le x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3: $\boxed{(a)}$ $E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\, dx\, dy$

We want to replace $f_{X,Y}(x,y)$

by 2. To do that we need to consider the region

$$\{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x+y \le 1\}$$

$$= \{(x,y) : 0 \le y \le 1,\ 0 \le x \le 1-y\}$$

Hence,

$$E[X] = \int_0^1 \int_0^{1-y} x \cdot 2 \, dx \, dy$$

$$= \int_0^1 (1-y)^2 \, dy$$

$$= \frac{1}{3}.$$

$\boxed{(b)}$ Similarily,

$$E[X+Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) f_{X,Y}(x,y) \, dx \, dy$$

$$= \int_0^1 \int_0^{1-y} (x+y) \cdot 2 \, dx \, dy$$

$$= \int_0^1 (1-y)^2 \, dy + \int_0^1 2y(1-y) \, dy$$

$$= \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}.$$

$\boxed{(c)}$ $V(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)^2 f_{X,Y}(x,y)\,dx\,dy - \left(\frac{2}{3}\right)^2$$

$$= \int_0^1\int_0^{1-y} (x+y)^2 . 2\,dx\,dy - \frac{4}{9}$$

$$= \int_0^1 \frac{2(1-y)^3}{3}\,dy + \int_0^1 2y^2(1-y)\,dy + \int_0^1 4\,\frac{y(1-y)^2}{2}\,dy - \frac{4}{9}$$

$$= \frac{2}{3}\times\frac{1}{4} + \frac{1}{6} + \frac{1}{6} - \frac{4}{9}$$

$$= \frac{1}{18}.$$

Problem 4: $\boxed{(a)}$ Since X and Y are independent,

$$\mathrm{Cov}(X,Y) = 0.$$

$\boxed{(b)}$ $V(X+Y) = V(X) + V(Y) + 2\,\mathrm{Cov}(X,Y)$

$$= 16 + 4 + 2\times 0$$

$$= 20.$$

$\boxed{(c)}$ Note that $V(X-Y) = V(1 \times X + (-1) \times Y)$

$$= (1)^2 V(X) + (-1)^2 V(Y) + 2 \times 1 \times (-1)\, Cov(X, Y)$$

$$= V(X) + V(Y) - 2\, Cov(X, Y).$$

Hence,

$$V(X-Y) = V(X) + V(Y) - 2\,Cov(X, Y)$$
$$= 16 + 4 - 2 \times 0$$
$$= 20.$$