Seshadri Constants, Fake Projective Planes, and Related Topics

Luca F. Di Cerbo

University of Florida

Surfaces quotient de la boule unité et réseaux, CIRM, February 26, 2019

Discussion will mention results from

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The Toledo Invariant, and Seshadri Constants of Fake Projective Planes

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Most recent results joint with

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Gennaro Di Brino
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- Ideally, inspire you to find this Stuff Interesting;
- Finally, I want to discuss applications of this circle of ideas to Exceptional Collections and Bicanonical Maps on Fake Projective Planes.

Introduction

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Let X be a smooth projective variety (surface) and L a Nef (ample) line bundle on X. Then

$$\epsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{mult_x C},$$

where the infimum is taken over all curves $C \subset X$ containing the point x, is the Seshadri Constant of L at $x \in X$. Finally

$$\epsilon(L) := \inf_{x \in X} \epsilon(L, x)$$

is the Global Seshadri Constant of L.

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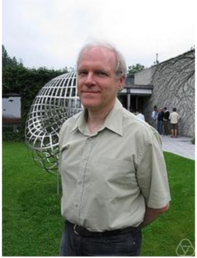

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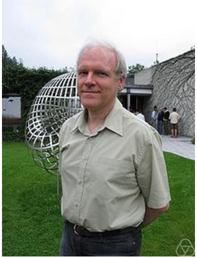

JEAN-PIERRE DEMAILLY (1957–), **Stefan Bergman Prize** in 2015.

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Let X be a smooth projective variety and L a line bundle on X. Then L is **Ample** *if and only if there is $\delta > 0$ such that*

$$\frac{L \cdot C}{mult_x C} \geq \delta$$

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$$L \text{ is } \textbf{Ample} \iff \epsilon(L) \geq \delta > 0$$

Seshadri Constants on Algebraic Surfaces

Theorem (Ein-Lazarsfeld, 1993)

Let S be a smooth projective surface and L an **ample** *line bundle on S. Then*

$$\epsilon(L, x) \geq 1,$$

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Theorem (Miranda, 1994)

For any $\delta > 0$, there exists a surface S, a point $x_0 \in S$ and an **Ample** *line bundle L on S such that*

$$\epsilon(L, x_0) < \delta.$$

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$\epsilon(L) \geq \epsilon_0$ for any **ample** line bundle L on S!

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Let S be a smooth projective surface with **Pic**$(S) = 1$, *and let L be an* **ample** *line bundle on S. For any point $x \in S$ we have*

- $\epsilon(L, x) \geq 1$ *if S is* **not** *if general type;*
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- Rest of the proof **Case-by-Case Analysis**

Seshadri Costants of Fake Projective Planes

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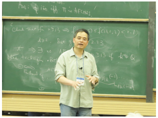

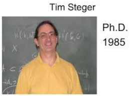

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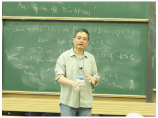

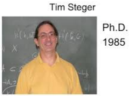


Definition

A **Fake Projective Plane** *is a surface of general type* S *with* $c_2(S) = 3$ *and* $p_g = h^0(S; K_S) = 0$.

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Definition

For a **Fake Projective Plane** S*, we denote by* L_1 *any ample generator of the torsion free part of* **Pic**(S)*. Similarly, for any* $k \geq 1$ *we set* $L_k = L_1^{\otimes k}$

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and C^* is the smooth locus of C

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In conclusion

Proposition

Let X be a complex hyperbolic surface. Given a reduced irreducible curve $C \subset X$, let us denote by $\overline{C}$ its normalization. We then have

$$0 < K_X \cdot C \leq 3(g(\overline{C}) - 1)$$

with equality if and only if C is an **immersed totally geodesic curve**.

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Corollary

Let C be a reduced irreducible curve in a fake projective plane S numerically equivalent to L_k for some $k \geq 1$. Let $\overline{C}$ be its normalization, we then have $g(\overline{C}) > 1 + k$.

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Note that if $C \equiv L_1 \implies C$ is **smooth** with $g(C) = 3$.

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Nevertheless, for our purposes, we just need the following:

Proposition

Let C be a reduced, irreducible singular curve in a fake projective planes S. Let $C \equiv L_k$ for some $k \geq 2$. For any **singular** *point $x \in C$, we have $2 \leq m_x \leq k$, where m_x denotes the multiplicity of x.*

Proof

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$$\delta_x = \dim_{\mathbb{C}}(i_* \mathcal{O}_{\overline{C}} / \mathcal{O}_C)_x$$

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Also... and a Question

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Question

Can we explicitly compute the Seshadri Constants of fake $\mathbb{P}^4_{\mathbb{C}}$*'s?*

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The study of curves C inside fake projective planes S gives:

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- Results concerning **Exceptional Collections** on S

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Let S be a fake projective plane. The 2-canonical map

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Take $L = \mathcal{O}_S(1)$ where $\mathcal{O}_S(1)$ is the **unique** G_{21}-equivariant line bundle

Definition

Let S be a fake projective plane, and let L be an ample generator of $Pic(S)$. Then the sequence $(\mathcal{O}_S, -L, -L^{\otimes 2})$ is **exceptional** *if and only if*

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such that $K_S \cong \mathcal{O}_S(1)^{\otimes 3}$

Theorem (with Di Brino)

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for **any** torsion line bundle T**!**

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Thanks for listening!

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