Higher Graphs Manifolds and Singer Conjecture

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Singer Conjecture:

- (M^n, g) closed aspherical $\Longrightarrow b_i^{(2)}(M) = 0 \quad i \neq n/2$

top. univ. cover $\tilde{M}$ is contractible $\quad M = \Gamma \backslash \tilde{M} \qquad \tilde{g} := \pi^* g$

$\pi : \tilde{M} \to M \qquad \pi : (\tilde{M}, \tilde{g}) \to (M, g)$

$$b_i^{(2)}(M) := \dim_\Gamma \mathcal{H}_2^i(\tilde{M})$$

$$\mathcal{H}_2^i(\tilde{M}) := \left\{ \omega \in \Omega^i(\tilde{M}) \;\middle|\; \Delta_d \omega = 0, \int_{\tilde{M}} \omega \wedge * \omega < \infty \right\}$$

$\Delta_d = dd^* + d^*d$

<u>Fact</u> $\quad b_i^{(2)}(M) = 0 \Longleftrightarrow \mathcal{H}_2^i(\tilde{M}) \cong 0$

<u>Remark</u> The $b_i^{(2)}(M)$ are homotopy invariant (Atiyah – Dodziuk)

and $\chi(M) = \sum_{i=0}^{\dim M} (-1)^i b_i^{(2)}(M)$

Thm

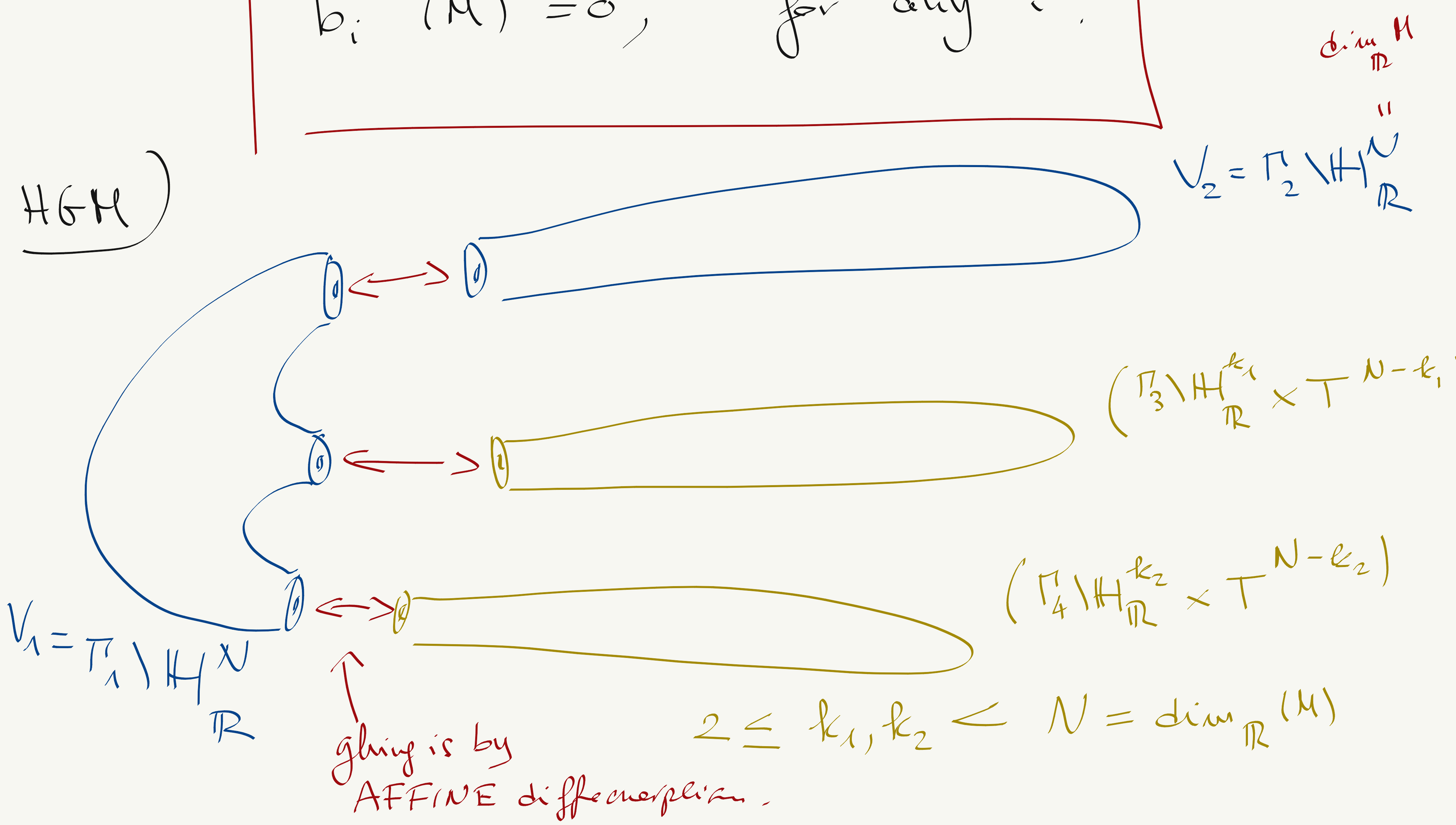
Let M be a higher graph manifold (HGM) with $k \geq 1$ pure real-hyperbolic pieces say $\{(V_j, g_{-1})\}_{j=1}^{k}$ and $\pi_1(M)$ residually finite (RF). If $\dim_{\mathbb{R}}(M) = 2n$, we have

$$\boxed{b_i^{(2)}(M) = \begin{cases} (-1)^n \chi_{top}(M) = (-1)^n \sum_{j=1}^{k} \chi_{top}(V_j) & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}}$$

Finally, if $\dim_{\mathbb{R}}(M) = 2n+1$, we have

$$\boxed{b_i^{(2)}(M) = 0, \quad \text{for any } i.}$$

HGM)



Why the assumption on $\pi_1(M)$ to be RF?

Because the proof uses Lück's approximation theorem

<u>Lück</u> (and def. of RF)

$\Lambda := \pi_1(M) \Longrightarrow \exists \{\Lambda_k\}$ sequence of nested subgroups

s.t. $\Lambda_k \trianglelefteq \Lambda$, $[\Lambda_k : \Lambda] < \infty$, $\bigcap_k \Lambda_k = id$ then

normal

$$\boxed{\lim_{k \to \infty} \frac{b_i(M^k)}{\deg(\pi_k)} = b_i^{(2)}(M)} \qquad (1)$$

where $\pi_k : M^k \longrightarrow M$ is the regular cover associated to Λ_k.

<u>Remark</u>

Thanks to Lück, in order to prove Singer it suffices to show that the limit in (1) is zero for $i < (N-1)/2$.

<u>Remark</u>

For HGM, if $k=0$ (no pure-real hyperbolic pieces)
Singer is known
Gromov (with RF)
Sauer (in general no RF)

Price Inequalities for Harmonic Forms in a nutshell.

$$(M^N, g) \qquad inj_{(M,g)} \geq \varepsilon > 0, \quad |Sec_g| \leq B$$

i-th Betti number density.

$$b_i(M) = \int_M \rho_{b_i}(x)\, d\mu_g$$

and

$$\boxed{\begin{array}{l} \cdot \text{ We have} \\ \qquad 0 \leq \rho_{b_i}(x) \leq C(N,i) \\ \text{for any } x \in M; \end{array}} \qquad (2)$$

(3) $\cdot$ If $x \in M$ is such that $B_R(x)$ isometric to a ball in $(\mathbb{H}^N_R, g_{-1})$ then

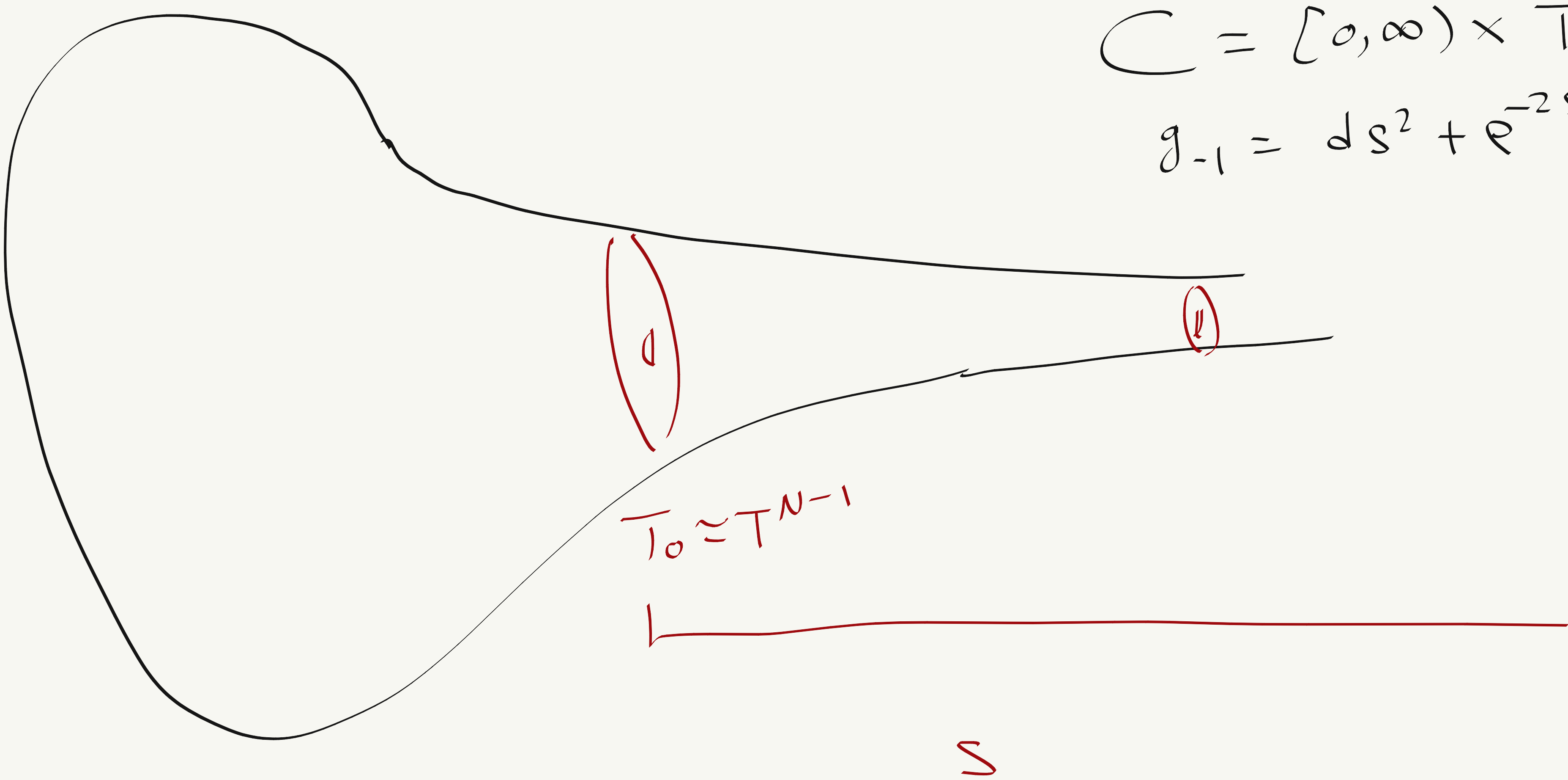
$$0 \leq \rho_{b_i}(x) \leq \begin{cases} D_1(N,i)\, e^{-(N-1-2i)R} & \text{if } (N-1-2i) > 0; \\ \dfrac{D_2(N,i)}{R} & \text{if } N-1-2i = 0. \end{cases}$$

Proof in a particular case.

$K = \Gamma \backslash \mathbb{H}^N_{\mathbb{R}}$ finite volume toral cusp

$C = [0,\infty) \times T^{N-1}$

$g_{-1} = ds^2 + e^{-2s} g_{T^{N-1}}$



$K \Longrightarrow \overline{K}$ manifold with boundary obtained by chopping the cusps.

Given two copies of $\overline{K}$ I construct a closed manifold $M = \overline{K} \#_{\psi} \overline{K}$

twisted double

affine

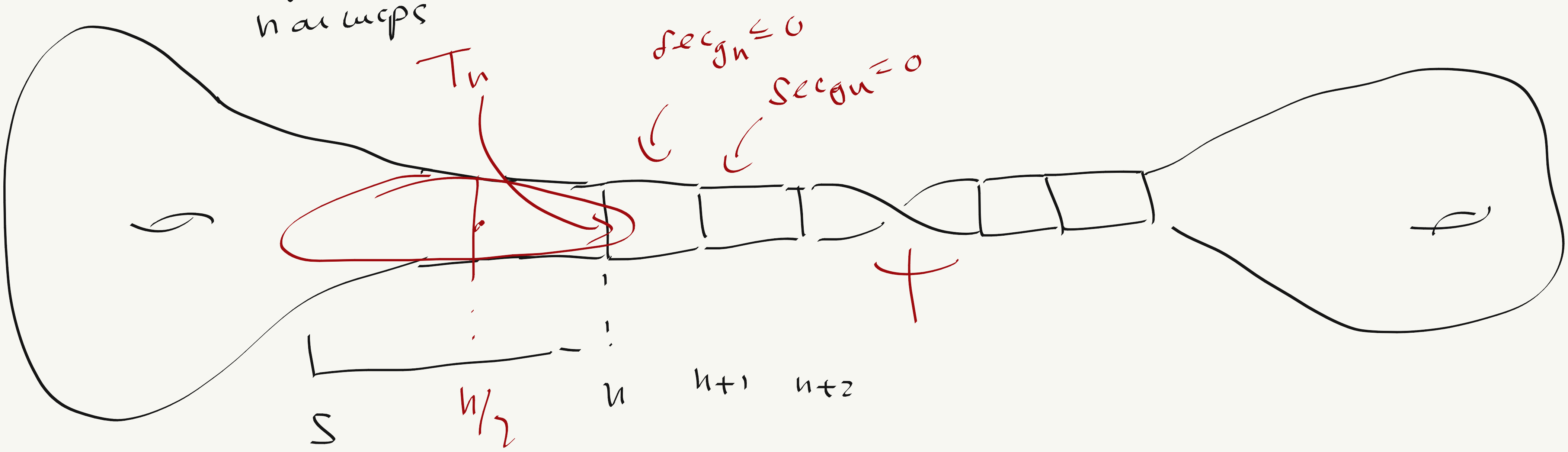
Example

$N=3$ $\quad T_0 = \mathbb{R}^2/\mathbb{Z}^2$ $\quad \mathrm{id} : T_0 \circlearrowleft$, $\quad A \in GL(2,\mathbb{Z})$

$\varphi_A : T_0 \circlearrowleft$ $\quad M = \overline{K} \#_{\varphi_A} \overline{K}$ $\quad$ e.g. $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Next I want to equip $M = \overline{K} \underset{+}{\#} \overline{K}$ with a sequence of metrics $\{g_n\}$ as follows:

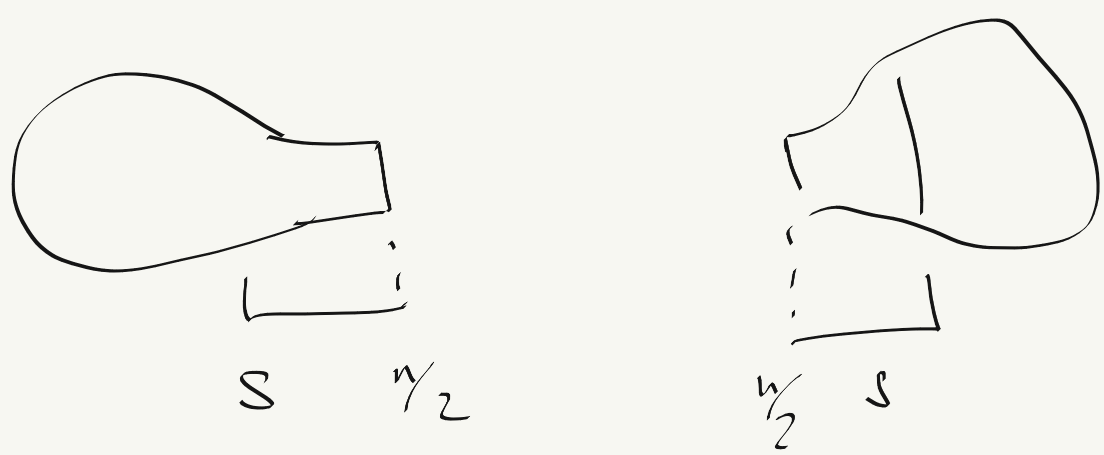
$g_n = g_{-1}$ up to height n and caps



s.t. $|\sec_{g_n}| \leq B$ for any n and

$$\boxed{\lim_{n \to \infty} \mathrm{Vol}_{g_n}(M) = 2\,\mathrm{Vol}_{g_{-1}}(N)} \qquad (4)$$

• $M_n =$



notice ~~that~~

$$\boxed{\begin{gathered}\mathrm{Vol}_{g_n}(M_n) = \mathrm{Vol}_{g_n}(M) - \varepsilon(n) \\ \lim_{n \to \infty} \varepsilon(n) = 0\end{gathered}} \qquad (5)$$

Say $\Lambda := \pi_1(M)$ is RF $\Rightarrow$ $\{\Lambda_k\}$ as in def of RF

$$(M^k, g_n^k) \xrightarrow{\text{Riemannian Regular Cover}} (M, g_n)$$

where $g_n^k := \pi_k^* g_n$. Finally, let's define

$$M_n^k := \pi_k^{-1}(M_n) \subset M^k.$$

By studying $(M^k, g_n^k) \xrightarrow{k \to \infty} (\tilde{M}, \tilde{g}_n)$, $\tilde{g}_n = \pi^* g_n$

one can show that $\exists$ $k_0 = k_0(n)$ s.t.

for any $k \geq k_0(n)$ and $p \in M_n^k$, there exists a ball centered at p of radius $n/2$ isometric to a ball $B_{n/2}(-)$ inside $(\mathbb{H}_{\mathbb{R}}^N, g_{-1})$.

Now for $i \leq \frac{N-1}{2}$

$$b_i(M^k) = \int_{M^k} \rho_{b_i}\, d\mu = \int_{M^k_n} \rho_{b_i}\, d\mu + \int_{M^k \setminus M^k_n} \rho_{b_i}\, d\mu$$

$$\leq \begin{cases} D_1(N,i)\, e^{-(N-1-2i)\frac{n}{2}} \operatorname{Vol}_{g_n^k}(M_n^k) + C(N,i) \operatorname{Vol}_{g_n^k}(M^k \setminus M_n^k) \\[2em] \dfrac{2 D_2(N,i)}{n} \operatorname{Vol}_{g_n^k}(M_n^k) + C(N,i) \operatorname{Vol}_{g_n^k}(M^k \setminus M_n^k) \end{cases}$$

cf. (2) and (3).

Then

$$\boxed{\frac{b_i(M^k)}{\operatorname{Vol}_{g_n^k}(M^k)} \leq f(n) + \frac{C(N,i)\, \varepsilon(n)}{\operatorname{Vol}_{g_n}(M)}}$$

where $f(n) \xrightarrow{n \to \infty} 0$

$\varepsilon(n) \xrightarrow{n \to \infty} 0$

But now $\operatorname{Vol}_{g_n^k}(M^k) = \deg(\pi_k) \operatorname{Vol}_{g_n}(M)$

$$\operatorname{Vol}_{g_n}(M) \xrightarrow{n \to \infty} 2 \operatorname{Vol}_{g_{-1}}(K)$$

$$\overline{\lim_{k\to\infty}} \frac{b_i(M^k)}{\deg(\pi_k)} = \lim_{k\to\infty} \frac{b_i(M^k)}{\deg(\pi_k)} = 0 = b_i^{(2)}(M)$$

* From previous page you see that

$$\overline{\lim_{k\to\infty}} \frac{b_i(M^k)}{\deg(\pi_k)} \leq f'(n) + C(N,i)\,\varepsilon'(n)$$

$$f'(n) \xrightarrow{n\to\infty} 0$$
$$\varepsilon'(n) \xrightarrow{n\to\infty} 0$$

$b_i^{(2)}(M) = 0$ for $i \leq \frac{N-1}{2}$, then Poincaré duality and the fact that

$$\chi(M) = 2\chi(K) \quad (\text{Mayer-Vietoris})$$

gives you the final result.