Extended Graph Manifolds, Dehn Fillings, and Einstein Metrics

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Geometry & Topology Seminar,
Mathematics Department,
Stony Brook University, April 26, 2022

Discussion will be based on results from

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- Show that Complex-Hyperbolic Einstein Dehn Filling **cannot** be performed in Dimension Four;
- Ideally, inspire you to find this topic interesting!

Introduction: Metrics, Distance, and Geodesics

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Definition

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Given a smooth path in a Riemannian manifold (M, g)

$$\alpha : [a, b] \to M,$$

we define its length by setting

$$L(\alpha) = \int_a^b g(\alpha'(t), \alpha'(t))^{1/2} dt, \quad \alpha'(t) \in T_{\alpha(t)}M.$$

We then define the distance between $p_1, p_2 \in M$, denoted by $d(p_1, p_2)$, to be the infimun for L over all smooth paths joining p_1 and p_2.

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Following the geodesics starting at a point $p \in M$, we obtain the so-called **exponential map**:

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It defines local coordinates around $p \in M$ known as Geodesic Normal Coordinates.

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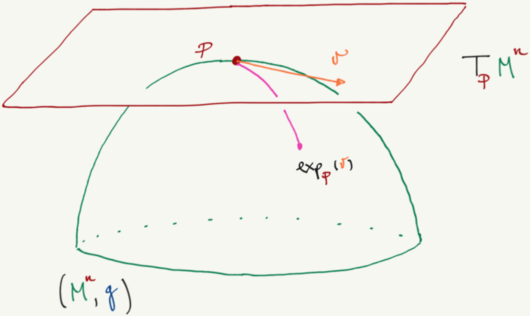

The **expansion** of the **Volume Element** in Geodesic Normal Coordinates and is given by:

$$d\mu_g = \left(1 - \frac{1}{6} Ric_{ij} x^i x^j + O(|x|^3)\right) dx^1 \wedge \ldots \wedge dx^n$$

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Volumes, Ricci Curvature, and Einstein Metrics

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Definition

A Riemannian metric is said to be **Einstein** *if its Ricci tensor satisfies*

$$Ric_g = \lambda g,$$

where the constant $\lambda \in \mathbb{R}$ is known as the cosmological or Einstein constant.

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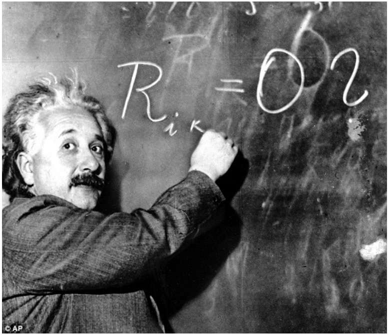

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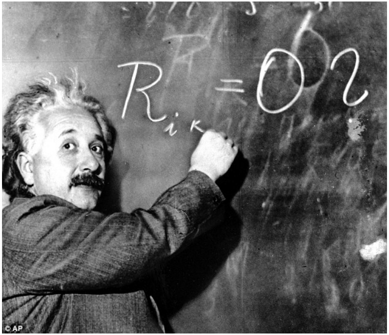

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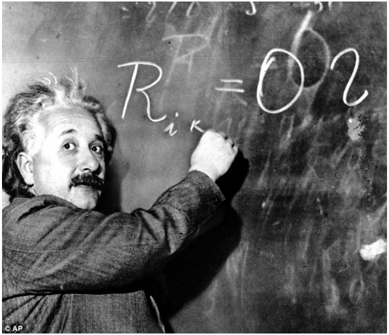

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And of course, he was exploring the **Lorentzian** case...

Classical Examples of Einstein Metrics

Einstein metrics always exist in dimension $n = 2$. Indeed, we have

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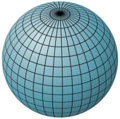

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With that said, these examples of Einstein manifolds are all locally symmetric and as such a bit **boring** from a local geometry point of view! Indeed, around each point they look exactly the same...

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S.-T. Yau (1949 –), **Fields Medal** in 1982.

Theorem (Yau)

- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda = 0$ (Ricci-flat) if and only if $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;*
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Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

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- **Hopefully** many more to come in my life time!!

Obstructions to the Existence of Einstein Metrics

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* Picture courtesy of the Archives of the Mathematisches Forschungsinstitut Oberwolfach

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Similarly, recall that the Signature $\sigma(M^4)$ of closed orientable 4-manifold is given as the signature of the natural bilinear form

$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$$

defined by counting **intersections with signs**.

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

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Combining these identities we obtain the desired inequality! □

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- $S^1 \times S^3$ (M. Berger). Notice that

$$\chi(S^1 \times S^3) = \sigma(S^1 \times S^3) = 0,$$

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- **Many more...**

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Extended Graph 4-Manifolds

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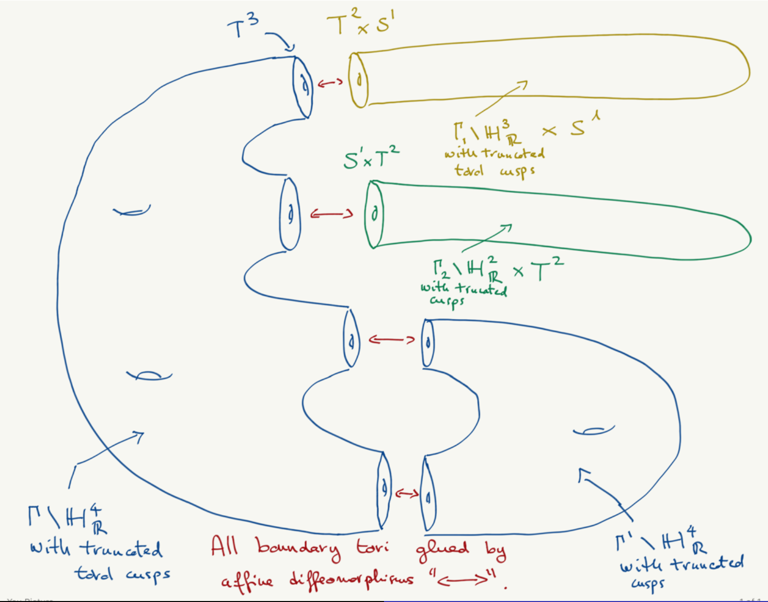

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Notice we always have **more than** one piece!

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- Notice that Closed Extended graph 3-manifolds do **NOT** support Einstein metrics
- Indeed, $Ric_g = \lambda g$ implies **constant sectional curvature** in dimension $n = 3$!
- This theorem then shows that graph-like manifolds carry over their **aversion** to Einstein metrics from dimension three to four.

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- The study of Einstein metrics on manifolds of dimension $n \geq 5$ remains rather obscure when compared to dimension $n = 4$.
- In fact, **no uniqueness or non-existence results** are currently known in higher dimensions!
- Maybe someone in the audience, perhaps a student, will take on the challenge!

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Lemma (Improved Hithin-Thorpe Inequality due to LeBrun, 1999)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ, Euler characteristic χ, and $\lambda < 0$. Then

$$2\chi(M) - 3|\sigma(M)| \geq \frac{3}{2\pi^2} Vol_{Ric}(M)$$

where the minimal Ricci volume $Vol_{Ric}(M)$ is defined as

$$Vol_{Ric}(M) := inf_g\{Vol_g(M) \quad | \quad Ric_g \geq -3g\}.$$

Moreover, equality occurs if and only if g is **half-conformally flat** *and it realizes the minimal Ricci volume (up to scaling). Finally, if $\sigma(M) = 0$ and the equality is achieved, then M is real-hyperbolic.*

Lemma

Let M be an extended graph 4-manifold **without** *pure pieces. We have* $\chi(M) = \sigma(M) = 0$. *If M has $k \geq 1$ pure real-hyperbolic pieces say* $(V_i := \Gamma_i \backslash \mathbb{H}^4_{\mathbb{R}}, g_{-1})_{i=1}^{k}$, *we then have*

$$\chi(M) = \sum_{i=1}^{k} \chi(V_i) > 0, \quad \sigma(M) = 0.$$

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Lemma (Connell-Suaréz-Serrato, 2019)

Let M be an extended graph 4-manifold **with** *$k \geq 1$ pure real-hyperbolic pieces say $(V_i, g_{-1})_{i=1}^k$, we then have*

$$Vol_{Ric}(M) = \sum_{i=1}^{k} Vol_{g_{-1}}(V_i) = \frac{4\pi^2}{3} \sum_{i=1}^{k} \chi(V_i) = \frac{4\pi^2}{3} \chi(M).$$

Concluding, we have two **distinct** cases:

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Anderson's Einstein Dehn Filling Theory in a Nutshell

- Start with a finite volume real-hyperbolic manifold $(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}, g_{-1})$ with torus cusps.

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- Truncate appropriately the cusps to obtain a manifold with **flat** torus boundaries $(T^{n-1}, g_{Euc.})$, say N.
- Any simple closed geodesic $\sigma\in(T^{n-1}, g_{Euc.})$ can be written as

$$[\sigma]=\sum_{i=1}^{n-1}\sigma^i[v_i]$$

where $\mathbb{Z}^{n-1}=span_{\mathbb{Z}}\{v_1,...,v_{n-1}\}$

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- Obtain a closed manifold

$$M_{\sigma_1,\dots,\sigma_k} := [\cup_{j=1}^{k}(D_j^2 \times T_j^{n-2})] \bigcup_{\phi_1,\dots,\phi_k} N,$$

where k is the **number** of cusps of $(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}, g_{-1})$.

Pictorial Real-Hyperbolic Dehn Filling

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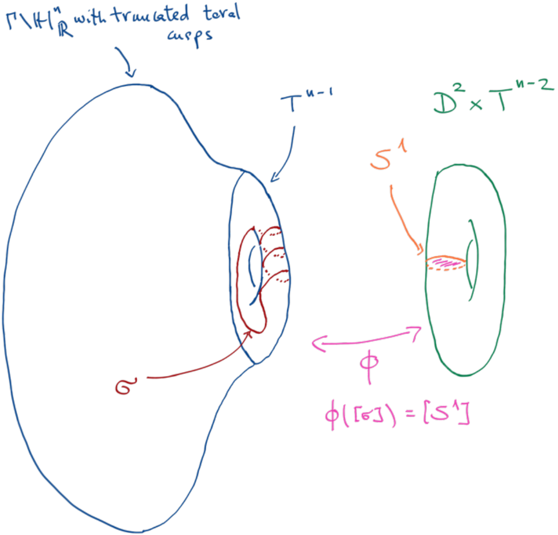

Theorems for Real-Hyperbolic Dehn Filled Manifolds

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Theorem (Gromov-Thurston)

Let $(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}, g_{-1})$ *be a finite volume real-hyperbolic manifold with* $k \geq 1$ *toral cusps and dimension* $n \geq 3$*. The* **Dehn filled** *manifold* $M_{\sigma_1,...,\sigma_k}$ *admits a metric of non-positive sectional curvature if*

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What about the existence of Einstein metrics?

Theorem (Michael Anderson, 2006)

$M_{\sigma_1,...,\sigma_k}$ admits an Einstein metric with negative cosmological constant if

$$l(\sigma_j) >> 1, \quad j = 1, ..., k.$$

Question (M. Anderson)

Can we generalize this theory to the Complex-Hyperbolic setting (especially) in dimension four?

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M. Anderson, *A survey of Einstein metrics on 4-manifolds*, **Handbook of geometric analysis** No. 3, Adv. Lect. Math. (ALM), vol.14, Int. Press, Sommerville, MA, 2010, pp. 1-39.

Preliminaries: Complex-Hyperbolic n-Manifolds

Let us denote by $\mathbb{B}^n$ the unit ball in $\mathbb{C}^n$

$$\boxed{\mathbb{B}^n := \{|z_1|^2 + ... + |z_n|^2 < 1, \quad (z_1, ..., z_n) \in \mathbb{C}^n\}}$$

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The Bergman metric ω_B is invariant under the action of $\mathrm{PU}(n,1)$. In fact, $\mathrm{PU}(n,1)$ is the group of holomorphic isometries of $(\mathbb{B}^n, \omega_B)$

Concluding, one can show that

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So that, any *Complex-Hyperbolic n-Manifold* X is given by the quotient

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where Γ is **torsion free lattice**. For this reason, these spaces are often just called *Ball Quotients*.

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If the action of Γ is co-compact $\Rightarrow$ $\Gamma\backslash X$ is a compact complex-hyperbolic manifold

If Γ is non-uniform $\Rightarrow$ $\Gamma\backslash X$ is an open complex space with finitely many cusps with **infra-nilmanifold cross sections**.

Topological Description of the Cusp Cross Sections in Dimension 4

- Their **topological universal cover** is the Heisenberg Group $\mathcal{H}^3$ (2-step nilpotent)

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- For $\mathbf{k} \geq 1$ integer, consider the lattice in $\mathcal{H}^3$ given by

$$\Lambda_k := \begin{bmatrix} 1 & m & p/k \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, \quad m, n, p, k \in \mathbb{Z} \quad (k \text{ is } \textbf{fixed}).$$

We have that any **3-nilmanifold** is diffeomorphic to $N_k := \Lambda_k \backslash \mathcal{H}^3$ for some k.

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- General **3-infra-nilmanifolds** are then simply *finite* quotients of the N_k's.

Complex-Hyperbolic Dehn Filling in Dimension Four

- Start with a finite volume

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- Truncate the cusps of X^4 in order to obtain a manifold with finitely many boundary components diffeomorphic to a subset of the N_k's.

Complex-Hyperbolic Dehn Filling in Dimension Four

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$$\boxed{X^4 = \Gamma\backslash\mathbb{B}^2, \quad \Gamma \leq \mathrm{PU}(2,1)}$$

  with **nilmanifolds** cusp cross sections.
- Truncate the cusps of X^4 in order to obtain a manifold with finitely many boundary components diffeomorphic to a subset of the N_k's.
- Recall that the N_k's do **not admit** orientation-reversing diffeomorphisms.

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- For each boundary component N_k let $\bar{D}_k$ be the associated disc bundle considered with the opposite orientation.
- Fill each boundary component N_k by gluing in $\bar{D}_k$ with an orientation-preserving diffeomorphism of N_k in order to get a closed oriented 4-manifold (the **Dehn filled** manifold).

Impossibility of Complex-Hyperbolic Einstein Dehn Filling

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Theorem (DC, Golla 2021)

For each positive integer k there is a complex-hyperbolic surface X_k with cusps all diffeomorphic to $N_k \times \mathbb{R}^+$ whose **Dehn filling** *compactification via the identity diffeomorphism on the boundaries, say $\tilde{X}_k$, satisfies*

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Can we use this construction to prove a general obstruction?

Corollary (DC, Golla 2021)

Complex-hyperbolic **Einstein Dehn** *filling cannot be performed in dimension four.*

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Let X_k be a complex-hyperbolic surface with cusps as in the previous theorem. By the additivity of the Euler characteristic, and Novikov additivity for the signature of manifolds with boundary, we know that any Dehn filling of X_k has the same **Euler characteristic** and **signature** as $\tilde{X}_k$.

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Proof.

Let X_k be a complex-hyperbolic surface with cusps as in the previous theorem. By the additivity of the Euler characteristic, and Novikov additivity for the signature of manifolds with boundary, we know that any Dehn filling of X_k has the same **Euler characteristic** and **signature** as $\tilde{X}_k$. More precisely, any such Dehn filling **must** have

$$\chi = |\sigma| > 0 \quad \Rightarrow \quad \textbf{No Einstein Metrics!}$$

□

Thanks for listening,

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and

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It was a great pleasure to visit Stony Brook!