Curvature, Macroscopic Dimensions, and Symmetric Products of Surfaces

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UNIVERSITY of FLORIDA

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Discussion will be based on results from

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joint with my colleague **A. Dranishnikov** and his student **E. Jauhari**.

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- Present a **detailed** study of the curvature and symplectic asphericity properties of Symmetric Products of Surfaces;

- Address the **Gromov-Lawson** and **Gromov Conjectures** in the Kähler projective setting and draw **new** connections between complex algebraic geometry and macroscopic dimensions.

Introduction: Metrics, Curvature, and Macroscopic Dimensions

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Given a smooth path in a Riemannian manifold (M, g)

$$\alpha : [a, b] \to M,$$

we define its length by setting

$$L(\alpha) = \int_a^b g(\alpha'(t), \alpha'(t))^{1/2} dt, \quad \alpha'(t) \in T_{\alpha(t)}M.$$

We then define the distance between $p_1, p_2 \in M$, denoted by $d(p_1, p_2)$, to be the infimun for L over all smooth paths joining p_1 and p_2.

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It defines local coordinates around $p \in M$ known as Geodesic Normal Coordinates.

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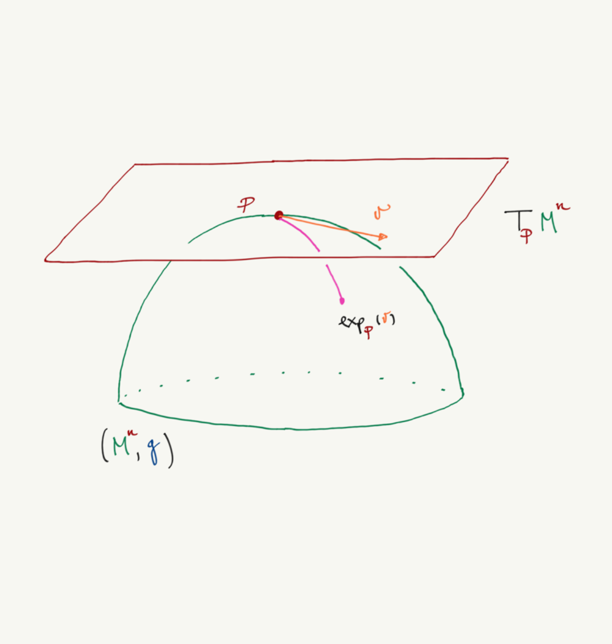

The **expansion** of the **Volume Element** in Geodesic Normal Coordinates and is given by:

$$d\mu_g = \left(1 - \frac{1}{6} Ric_{ij} x^i x^j + O(|x|^3)\right) dx^1 \wedge \ldots \wedge dx^n$$

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$$s_g : M \longrightarrow \mathbb{R}$$

can be defined for any $p \in M$ by the Taylor expansion

$$\boxed{\frac{Vol_g(B_p(R))}{Vol_{\mathbb{R}^n}(B_0(R))} = 1 - \frac{s_g(p)}{6(n+2)} R^2 + o(R^2)}$$

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Ok, what does this mean?!? At this stage, it seems we are simply (and optimistically!) generalizing the picture from the Cheeger-Gromoll splitting theorem under the **much** stronger non-negative Ricci curvature assumption.

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For a closed Riemannian manifold M*, its* **macroscopic dimension**, *denoted* $\dim_{mc} M$*, is the smallest integer* m *such that there exists a uniformly cobounded, proper, continuous map*

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- For example, if M is **compact** then $\dim_{mc} M = 0$.
- Similarly, if M is a Riemannian product with a compact factor, say S^k, we have:

$$\boxed{M^n = N^{n-k} \times S^k \quad \Rightarrow \quad \dim_{mc} M \leq n - k.}$$

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Question

Let M be a closed Riemannian manifol and let $\widetilde{M}$ be its Riemannian universal cover. Under which condition on M do we have the equality

$$\dim_{mc} \widetilde{M} = \dim_{MC} \widetilde{M}?$$

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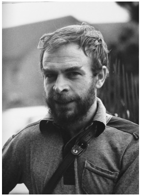

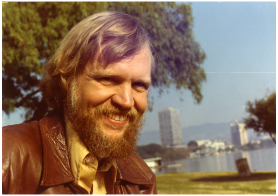

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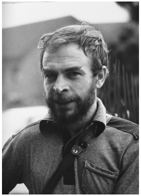

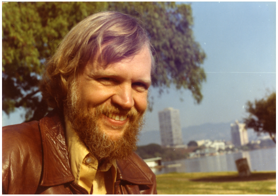

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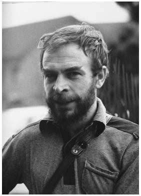
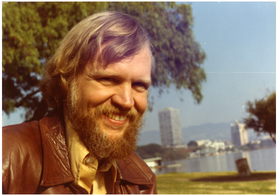

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*Author: Dirk Ferus. Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

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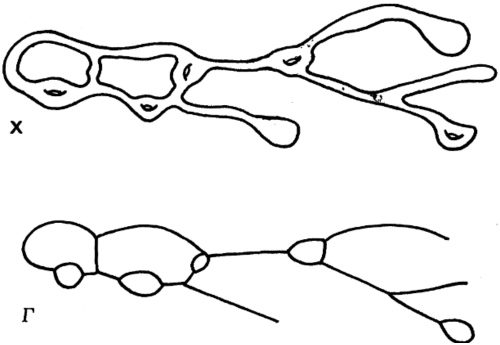
one finds the following tantalizing result:

Corollary 10.11. — *Let X be a closed 3-manifold of scalar curvature* ≥ 1. *Then there exists a distance decreasing map* $f : X \to \Gamma$ *onto a metric graph (a linear graph in* $\mathbf{R}^N$, *say), so that, for each* $p \in \Gamma$,

$$\text{diameter}(f^{-1}(p)) \leqq 12\pi.$$

Note. — This means that 3-manifolds with $\kappa \geq 1$ and diameter $\gg 1$ are "long and thin". Of course, arbitrarily long and complicated manifolds of this type can be constructed by taking connected sums (as in [GL_2]) of copies of $S^1 \times S^2$ and S^3.



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Theorem (DC-Dranishnikov-Jauhari, 2025)

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- **Many more results arise from the proof of this theorem...**

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***Definition**: A closed orientable n-manifold M is **rationally essential** if $u_*[M] \neq 0 \in H_n(B\pi_1(M); \mathbb{Q})$ where $u : M \to B\pi_1$ classifies the fundamental group.

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- Note that this **implication** still holds under the so-called Weak Gromov Conjecture $\dim_{mc} \widetilde{M} \leq n - 1$.

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Positive Results

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Theorem (DC-Dranishnikov-Jauhari, 2025)

- *If M^n admits a metric of non-negative Ricci curvature, we have* $\boxed{\dim_{mc} \widetilde{M} = \dim_{MC} \widetilde{M} \leq n}$ *with equality if and only if M^n is a flat n-manifold.*

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- *Assume there exists a degree one map $f : M \to N$ to a closed, orientable, closed n-manifold with $\dim_{mc} \widetilde{N} = n$ such that $f_* : \pi_1(M) \to \pi_1(N)$ is an isomorphism. We then have* $\boxed{\dim_{mc} \widetilde{M} = \dim_{MC} \widetilde{M} = n}$.

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- *For a given n-manifold M with $\dim_{mc} \widetilde{M} = n \geq 3$, we have* $\boxed{\dim_{mc} \widetilde{M\#N} = \dim_{MC} \widetilde{M\#N} = n}$ *for any closed n-manifold N.*

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- *For a given* n*-manifold* M *with* $\dim_{mc} \widetilde{M} = n \geq 3$*, we have* $\boxed{\dim_{mc} \widetilde{M \# N} = \dim_{MC} \widetilde{M \# N} = n}$ *for any closed* n*-manifold* N.
- $\boxed{\dim_{mc} \widetilde{M} = \dim_{MC} \widetilde{M}}$ *for a closed manifold* M^n *with* $n = 2, 3$.

Negative Results: $\dim_{mc} \neq \dim_{MC}$

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Theorem (DC-Dranishnikov-Jauhari, 2025)

Let M_g be a closed orientable surface of genus g, and let $\mathbf{SP}^n(M_g)$ *be the* **symmetric** *n-th power of M_g.*

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Recall that $\dim_{\mathbb{R}} \mathbf{SP}^n(M_g) = 2n!^*$

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- We have the **Abel-Jacobi map** $\mu_n : \mathbf{SP}^n(M_g) \to Jac(M_g)$ defined as

$$\mu_n(Q) := \left(\sum_\lambda \int_{p_0}^{p_\lambda} \omega_1, \ldots, \sum_\lambda \int_{p_0}^{p_\lambda} \omega_g \right), Q = \sum_{\lambda=1}^{n} p_\lambda, p_0 \in M_g$$

$\omega_1, ..., \omega_g$ is a **basis** for $H^0(M_g, K_{M_g})$.

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- More statements along these lines.
- Symmetric products of surfaces distinguish the two distinct notions of macroscopic dimension in a **dimensionally** and **geometrically** sharp way.

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A smooth projective variety X^n equipped with a **Kähler** *metric ω with $s_{g_\omega} > 0$ satisfies $\dim_{mc} \widetilde{X} \leq 2n - 2$.*

- Note that these conjectures **CRUCIALLY** add the **Kähler** condition on the metric. In particular, they do **NOT** imply the Gromov-Lawson & Gromov conjectures for Kähler manifolds.

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  On the curvature of complex surfaces, **Geom. Funct. Analysis** 5 (1995), 619-628.

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Let's dive into this!

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A smooth projective n-variety X with a birational morphism onto an **aspherical** *smooth projective n-variety Y* **cannot** *support a* **Kähler** *metric ω of positive scalar curvature. Moreover, we have*
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C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, *Existence of minimal models for varieties of general type*, **J. Amer. Math. Soc** 23 (2009), no. 5, 405-468.

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- This suggests that trying to solve the Gromov-Lawson conjecture by first solving the Gromov conjecture may be a **BAD** idea!
- Note that even in the **Kähler** setting, we only get a weak Gromov conjecture statement. Indeed, there are topological subtleties in showing that **macroscopic dimension** is a **birational** invariant.

- In particular, the proof relies upon some of the **LATEST** developments in the theories of the minimal model and positivity in algebraic geometry.
- This is in contrast with the **LOW-TECH** proof of the Kähler version of the Gromov-Lawson conjecture.
- This suggests that trying to solve the Gromov-Lawson conjecture by first solving the Gromov conjecture may be a **BAD** idea!
- Note that even in the **Kähler** setting, we only get a weak Gromov conjecture statement. Indeed, there are topological subtleties in showing that **macroscopic dimension** is a **birational** invariant.
- As a by-product of the proof of this theorem, we get $\boxed{\dim_{mc} = \dim_{MC}}$ for **Kähler** surfaces with positive scalar curvature. Our discovery that certain symmetric squares of curves have $\boxed{\dim_{mc} \neq \dim_{MC}}$ implies that this result does **NOT** extend to higher dimensions starting with threefolds!

Thanks for listening,

Thanks for listening,

and

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and

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and

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THE END