Seshadri Constants, Fake Projective Planes, and Related Topics

Luca F. Di Cerbo

University of Florida

Topology & Dynamics Seminar, University of Florida, March 10, 2020

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The Toledo Invariant, and Seshadri Constants of Fake Projective Planes

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Journal of the Mathematical Society of Japan
69 (2017), no.4, 1601–1610

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- Ideally, inspire you to find this Stuff Interesting;

- Finally, I want to discuss applications of this circle of ideas to Exceptional Collections and Bicanonical Maps on Fake Projective Planes.

Introduction

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Definition

Let X be a smooth projective variety (surface) and L a Nef (ample) line bundle on X. Then

$$\epsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{mult_x C},$$

where the infimum is taken over all curves $C \subset X$ containing the point x, is the Seshadri Constant of L at $x \in X$. Finally

$$\epsilon(L) := \inf_{x \in X} \epsilon(L, x)$$

is the Global Seshadri Constant of L.

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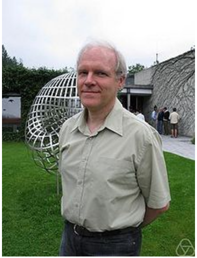

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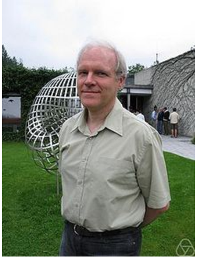

JEAN-PIERRE DEMAILLY (1957–), **Stefan Bergman Prize** in 2015.

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Theorem (Seshadri's Criterion for Ampleness)

Let X be a smooth projective variety and L a line bundle on X. Then L is **Ample** *if and only if there is $\delta > 0$ such that*

$$\frac{L \cdot C}{mult_x C} \geq \delta$$

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L is **Ample** $\iff$ $\epsilon(L) \geq \delta > 0$

Seshadri Constants on Algebraic Surfaces

Theorem (Ein-Lazarsfeld, 1993)

Let S be a smooth projective surface and L an **ample** *line bundle on S. Then*

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Theorem (Miranda, 1994)

For any $\delta > 0$, there exists a surface S, a point $x_0 \in S$ and an **Ample** *line bundle L on S such that*

$$\epsilon(L, x_0) < \delta.$$

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$\epsilon(L) \geq \epsilon_0$ for any **ample** line bundle L on S!

Theorem (Szemberg, 2008)

Let S be a smooth projective surface with $\textbf{Pic}(S) = 1$, *and let L be an* **ample** *line bundle on S. For any point $x \in S$ we have*

- $\epsilon(L, x) \geq 1$ *if S is* **not** *if general type;*
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- Rest of the proof **Case-by-Case Analysis**

Seshadri Costants of Fake Projective Planes

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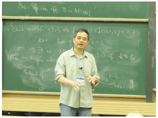

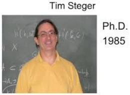


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  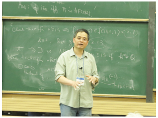 

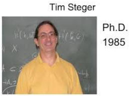

 

Definition

A **Fake Projective Plane** *is a surface of general type* S *with* $c_2(S) = 3$ *and* $p_g = h^0(S; K_S) = 0$.

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Definition

For a **Fake Projective Plane** S, *we denote by* L_1 *any ample generator of the torsion free part of* **Pic**(S). *Similarly, for any* $k \geq 1$ *we set* $L_k = L_1^{\otimes k}$

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and C^* is the smooth locus of C

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In conclusion

Proposition

Let X be a complex hyperbolic surface. Given a reduced irreducible curve $C \subset X$, let us denote by $\overline{C}$ its normalization. We then have

$$0 < K_X \cdot C \leq 3(g(\overline{C}) - 1)$$

with equality if and only if C is an **immersed totally geodesic curve**.

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Corollary

Let C be a reduced irreducible curve in a fake projective plane S numerically equivalent to L_k for some $k \geq 1$. Let $\overline{C}$ be its normalization, we then have $g(\overline{C}) > 1 + k$.

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Note that if $C \equiv L_1 \implies C$ is **smooth** with $g(C) = 3$.

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Nevertheless, for our purposes, we just need the following:

Proposition

Let C be a reduced, irreducible singular curve in a fake projective planes S. Let $C \equiv L_k$ for some $k \geq 2$. For any **singular** *point $x \in C$, we have $2 \leq m_x \leq k$, where m_x denotes the multiplicity of x.*

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$$\delta_x = \dim_{\mathbb{C}}(i_* \mathcal{O}_{\overline{C}} / \mathcal{O}_C)_x$$

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Question

Can we explicitly compute the Seshadri Constants of fake $\mathbb{P}^4_{\mathbb{C}}$*'s?*

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The study of curves C inside fake projective planes S gives:

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- Results concerning **Exceptional Collections** on S

Theorem (with Di Brino)

Let S be a fake projective plane. The 2-canonical map

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P. Pokora, H. Tutaj-Gasinska, *On Submaximal Curves in Fake Projective Planes*, **Arxiv:1910.06743v1[math.AG]**.

Exceptional Collections

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Let S be a fake projective plane, and let L be an ample generator of $Pic(S)$. Then the sequence $(\mathcal{O}_S, -L, -L^{\otimes 2})$ is **exceptional** *if and only if*

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Galkin, Katzarkov, Mellit, Shinder (Adv. Math. 2015) and Keum (2013 manuscript) discovered that Keum's fake projective planes have a **Standard Exceptional Collection**

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for **any** torsion line bundle T**!**

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I refer all of you to **Keum**'s ICM 2018 lecture

Thanks for Having Me!

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