Extended Graph Manifolds, and Einstein Metrics

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UF UNIVERSITY of FLORIDA

Colloquium,
Mathematics and Statistics Department,
UNC Greensboro, December 1, 2021

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- Prove a Non-Existence Theorem for Einstein metrics on Extended Graph 4-Manifolds;
- Ideally, inspire you to find this Stuff Interesting;
- Finally, I want to discuss some related open problems and possible generalizations to Higher Dimensions.

Introduction: Metrics, Distance, and Geodesics

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Given a smooth path in a Riemannian manifold (M, g)

$$\alpha : [a, b] \to M,$$

we define its length by setting

$$L(\alpha) = \int_a^b g(\alpha'(t), \alpha'(t))^{1/2} dt, \quad \alpha'(t) \in T_{\alpha(t)}M.$$

We then define the distance between $p_1, p_2 \in M$, denoted by $d(p_1, p_2)$, to be the infimun for L over all smooth paths joining p_1 and p_2.

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It defines local coordinates around $p \in M$ known as Geodesic Normal Coordinates.

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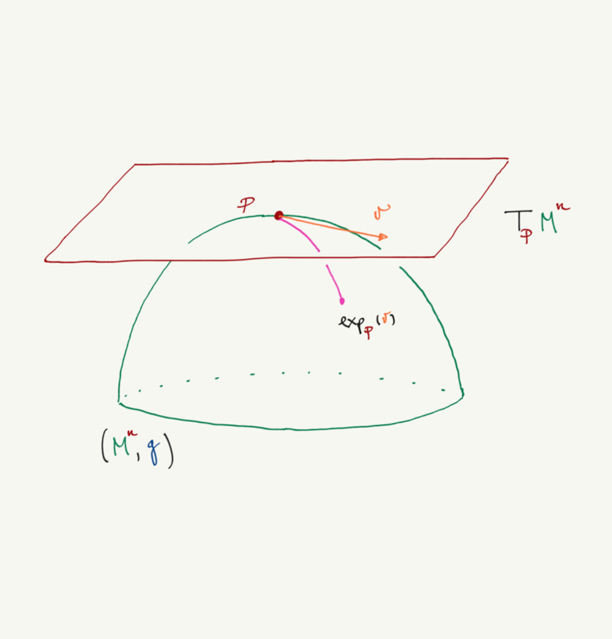

Volumes, Ricci Curvature, and Einstein Metrics

The **expansion** of the **Volume Element** in Geodesic Normal Coordinates and is given by:

$$d\mu_g = \left(1 - \frac{1}{6} Ric_{ij} x^i x^j + O(|x|^3)\right) dx^1 \wedge \dots \wedge dx^n$$

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A Riemannian metric is said to be **Einstein** *if its Ricci Tensor satisfies*

$$Ric_g = \lambda g,$$

where the constant $\lambda \in \mathbb{R}$ is known as the cosmological or Einstein constant.

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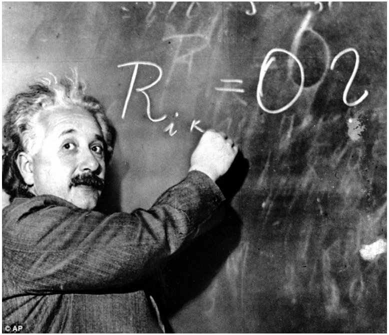

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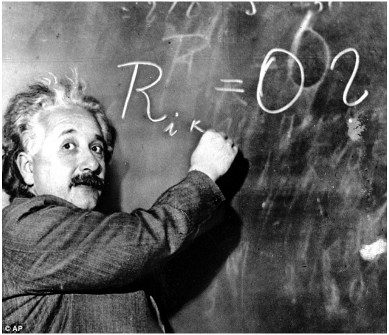

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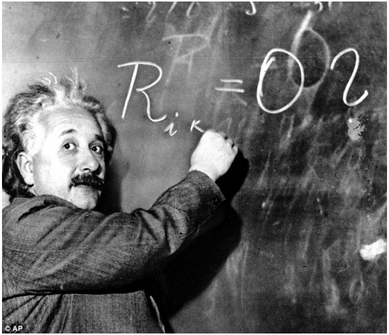

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And of course, he was exploring the **Lorentzian** case...

Classical Examples of Einstein Metrics

Einstein metrics always exist in dimension $n = 2$. Indeed, we have

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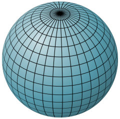

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With that said, these examples of Einstein manifolds are all locally symmetric and as such a bit **boring** from a local geometry point of view! Indeed, around each point they look exactly the same...

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S.-T. Yau (1949 –), **Fields Medal** in 1982.

Theorem (Yau)

- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda = 0$ (Ricci-flat) if and only if $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;*
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Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

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- **Hopefully** many more to come in my life time!!

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M. T. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, **J. Differential Geom. 73** (2006), no. 2, 219-261.

R. Bamler, *Construction of Einstein metrics by generalized Dehn filling*, **J. Eur. Math. Soc. (JEMS) 14** (2012), no. 3, 887-909.

J. Fine, B. Premoselli, *Examples of compact Einstein four-manifolds with negative curvature*, **J. Amer. Math. Soc. 33** (2020), no. 4, 991-1038.

Obstructions to the Existence of Einstein Metrics

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N. Hitchin (1946 –), **Shaw Prize** in 2016.

* Picture courtesy of the Archives of the Mathematisches Forschungsinstitut Oberwolfach

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Recall that the Euler characteristic is given by the alternating sum of **Betti numbers**

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Similarly, recall that the Signature of closed orientable 4-manifold is given as the signature of the natural bilinear form

$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$$

defined by counting **intersections with signs**.

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

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Combining these identities we obtain the desired inequality! □

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- **Many more...**

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Main Result, Conclusions, and Open Problems

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Extended Graph 4-Manifolds

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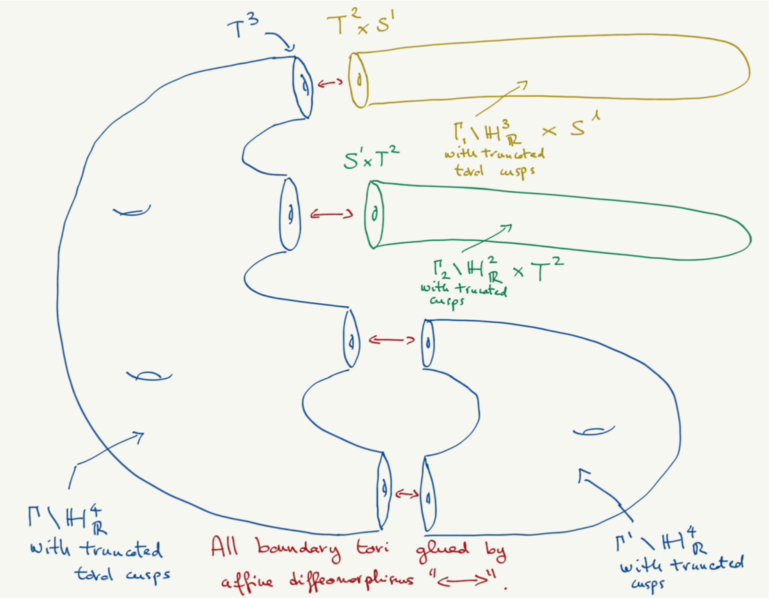

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Notice we always have **more than** one piece!

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Remarks and Comments

- Notice that Closed Extended graph 3-manifolds do **NOT** support Einstein metrics
- Indeed, $Ric_g = \lambda g$ implies **constant sectional curvature** in dimension $n = 3$!
- This theorem then shows that graph-like manifolds carry over their **aversion** to Einstein metrics from dimension three to four.

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- The study of Einstein metrics on manifolds of dimension $n \geq 5$ remains rather obscure when compared to dimension $n = 4$.
- In fact, **no uniqueness or non-existence results** are currently known in higher dimensions!
- Maybe someone in the audience, perhaps a student, will take on the challenge!

Sketch of the Proof

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Lemma (Improved Hithin-Thorpe Inequality due to LeBrun, 1999)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ, Euler characteristic χ, and $\lambda < 0$. Then

$$2\chi(M) - 3|\sigma(M)| \geq \frac{3}{2\pi^2} Vol_{Ric}(M)$$

where the minimal Ricci volume $Vol_{Ric}(M)$ is defined as

$$Vol_{Ric}(M) := inf_g\{Vol_g(M) \quad Ric_g \geq -3g\}.$$

Moreover, equality occurs if and only if g is **half-conformally flat** *and it realizes the minimal Ricci volume (up to scaling). Finally, if $\sigma(M) = 0$ and the equality is achieved, then M is real-hyperbolic.*

Lemma

Let M be an extended graph 4-manifold **without** *pure pieces. We have $\chi(M) = \sigma(M) = 0$. If M has $k \geq 1$ pure real-hyperbolic pieces say $(V_i := \Gamma_i \backslash \mathbb{H}^4_{\mathbb{R}}, g_{-1})_{i=1}^{k}$, we then have*

$$\chi(M) = \sum_{i=1}^{k} \chi(V_i) > 0, \quad \sigma(M) = 0.$$

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Lemma (Connell-Suaréz-Serrato, 2019)

Let M be an extended graph 4-manifold **with** *$k \geq 1$ pure real-hyperbolic pieces say $(V_i, g_{-1})_{i=1}^{k}$, we then have*

$$Vol_{Ric}(M) = \sum_{i=1}^{k} Vol_{g_{-1}}(V_i) = \frac{4\pi^2}{3} \sum_{i=1}^{k} \chi(V_i) = \frac{4\pi^2}{3} \chi(M).$$

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Here are the relevant papers for some of the lemmas:

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C. LeBrun, *Four-dimensional Einstein Manifolds, and Beyond*, **Surveys in differential geometry: essays on Einstein manifolds** 247-285, Surv. Differ. Geom., 6, *Int. Press, Boston, MA* , (1999).

C. Connell, P. Suaréz-Serrato, *On higher graph manifolds*, **Int. Math. Res. Not.** (2019), no. 5, 1281-1311.

G. Besson, G. Coutois, S. Gallot, *Entropies and rigidités des espaces localment symétriques de curbure strictment négative*, **Geom. Func. Anal. 5** (1995), 731-799.

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I hope to meet you all one day **in person**!