L^2-Type Asymptotic Invariants and Graph-Like Manifolds

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joint with M. Hull (NCS) (based on previous work with M. Stern (Duke))

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S.-T. Yau's List of "Open Problems in Geometry"

cpt, orientable, no boundary

Question 39

Let M^n be an n-dimensional closed manifold and M' be a compact (Galois) cover of M. Can one estimate the k-th Betti number b_k in terms of the order of the Galois group and the geometry of M? In particular, when does the normalized b_k tend to zero as the order of the Galois group tend to infinity? Perhaps this is true if $2k \neq n$ and M has negative curvature.

$\pi : M' \to M$, $\deg(\pi) < \infty$,

$$b_k(M') = \dim_{\mathbb{R}} H_k(M'; \mathbb{R})$$

and we set the normalized k-th Betti number to be

$$\frac{b_k(M')}{\deg(\pi)}$$

Connection between Yau's Question and Singer Conjecture is beautifully given by Lück Approximation Theorem

Singer Conjecture

(M^n, g) aspherical closed Riemannian manifold $\Longrightarrow$ $\boxed{b_i^{(2)}(M) = 0}$, $i \neq n/2$.

$\pi : \tilde{M} \to M$ top. universal cover

$\tilde{M}$ is contractible

$\pi : (\tilde{M}, \tilde{g}) \to (M, g)$

$\tilde{g} := \pi^* g$

$M = \pi \backslash \tilde{M}$

$$\boxed{b_i^{(2)}(M) = \dim_\pi \mathcal{H}_2^i(\tilde{M})}$$

$$\mathcal{H}_2^i(\tilde{M}) := \left\{ \omega \in \Omega^i(\tilde{M}) \;\middle|\; \Delta_d \omega = 0, \int_{\tilde{M}} \omega \wedge * \omega < \infty \right\}$$

$dd^* + d^*d$

Fact $\quad b_i^{(2)}(M) = 0 \iff \mathcal{H}_2^i(\tilde{M}) = 0,$

and moreover it can be proved that the $b_i^{(2)}$'s are homotopy invariant.

Lück Approximation Theorem (Gromov Conjecture)

Let (M^n, g) be as in Singer Conjecture with $\pi_1(M) := \Lambda$ residually finite. Given a sequence $\{\Lambda_k\}$ of nested subgroups such that

$$\boxed{\Lambda_k \lhd \Lambda \quad , \quad [\Lambda_k : \Lambda] < \infty , \quad \bigcap_k \Lambda_k = id}$$

then

$$\boxed{\lim_{k \to \infty} \frac{b_i(M^k)}{\deg(\pi_k)} = b_i^{(2)}(M)} \qquad (*)$$

where $\pi_k : M^k \longrightarrow M$ is the regular cover associated to Λ_k.

Remark Thanks to Lück, in order to prove the Singer Conjecture it suffices to show that the limit in $(*)$ is zero for $i \neq \dim_{\mathbb{R}} M/2$.

Goal of this Lecture is to give a POSITIVE answer to Yau's Question on a LARGE class of aspherical manifolds known as

$$\boxed{\begin{array}{l}\text{Extended Graph } n\text{-Manifolds} \\ \text{or} \\ \text{Higher Graph } n\text{-Manifolds}\end{array}}$$

Remark

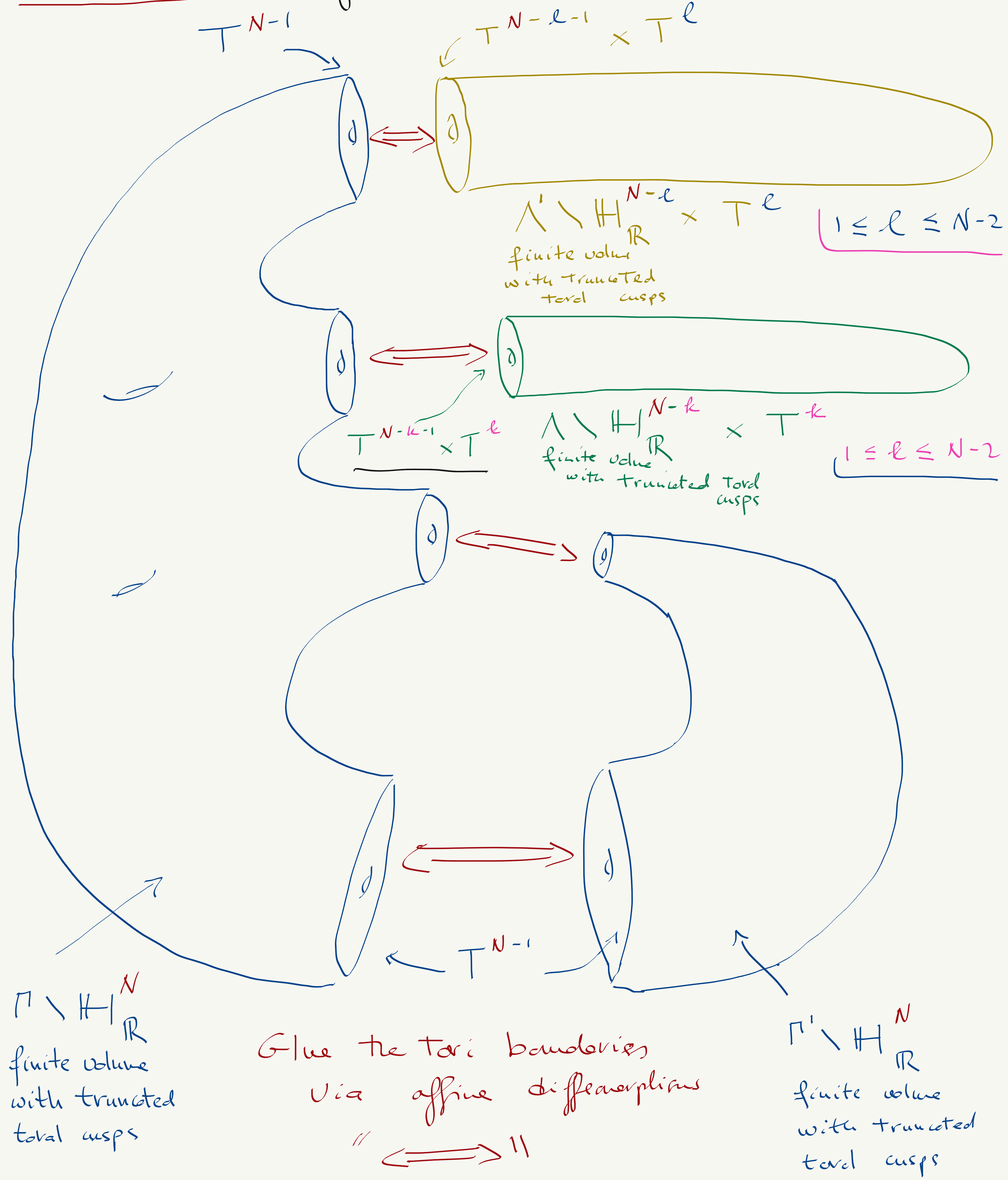
Extended Graph n-Manifolds were introduced in the book:

R. Frigerio, J.-F. Lafont, A. Sisto, Rigidity of High Dimensional Graph Manifolds, Astérisque Volume 372, 2015

See Also

C. Connell, P. Suaréz-Serrato, On higher graph manifolds, Int. Math. Res. Not. (2019), no. 5, 1281–1311

Pictorial Definition of Extended Graph N-manifolds



Glue the tori boundaries via affine diffeomorphisms

" $\Longleftrightarrow$ "