Theorem (joint with M. Hull)

Let M^N be an extended graph N-manifold with $\ell \geq 0$ pure real-hyperbolic pieces say $\{(V_j, g_{-1})\}_{j=1}^{\ell}$ and $\pi_1(M^N) = \Lambda$ residually finite. For any sequence $\{\Lambda_k\}$ of nested subgroups such that

$$\boxed{\Lambda_k \triangleleft \Lambda, \quad [\Lambda_k : \Lambda] < \infty, \quad \bigcap_k \Lambda_k = id}$$

we have

- N even

$$\boxed{\lim_{k\to\infty} \frac{b_i(M^k)}{\deg(\pi_k)} = \begin{cases} (-1)^{N/2}\chi_{top}(M) = (-1)^{N/2}\sum_{j=1}^{\ell}\chi_{top}(V_j), & i = N/2 \\ 0, & \text{if } i \neq N/2 \end{cases}}$$

- N odd

$$\boxed{\lim_{k\to\infty} \frac{b_i(M^k)}{\deg(\pi_k)} = 0, \quad \text{for any } i,}$$

where $\pi_k : M^k \longrightarrow M$ is the regular cover associated to Λ_k.

Remark Extended Graph N-manifolds do NOT have metrics with $\sec_g < 0$, and most of them not even with $\sec \leq 0$.

Price Inequalities for Harmonic Forms in a Nutshell (w/ Stern)

$$(M^N, g), \quad \boxed{\operatorname{inj}_{(M,g)} \geq \varepsilon > 0,} \quad \boxed{|\sec_g| \leq B}$$

$$b_i(M) = \int_M \rho_{b_i}(x)\, d\mu_g$$

density of i-th Betti number

and

$$\boxed{\bullet \quad 0 \leq \rho_{b_i}(x) \leq C(N, i), \text{ for any } x \in M;} \qquad (1)$$

dim M

degree

(2)

- If $x \in M$ is such that $B_R(x)$ is isometric to a ball in $(\mathbb{H}^N_{\mathbb{R}}, g_{-1})$ then

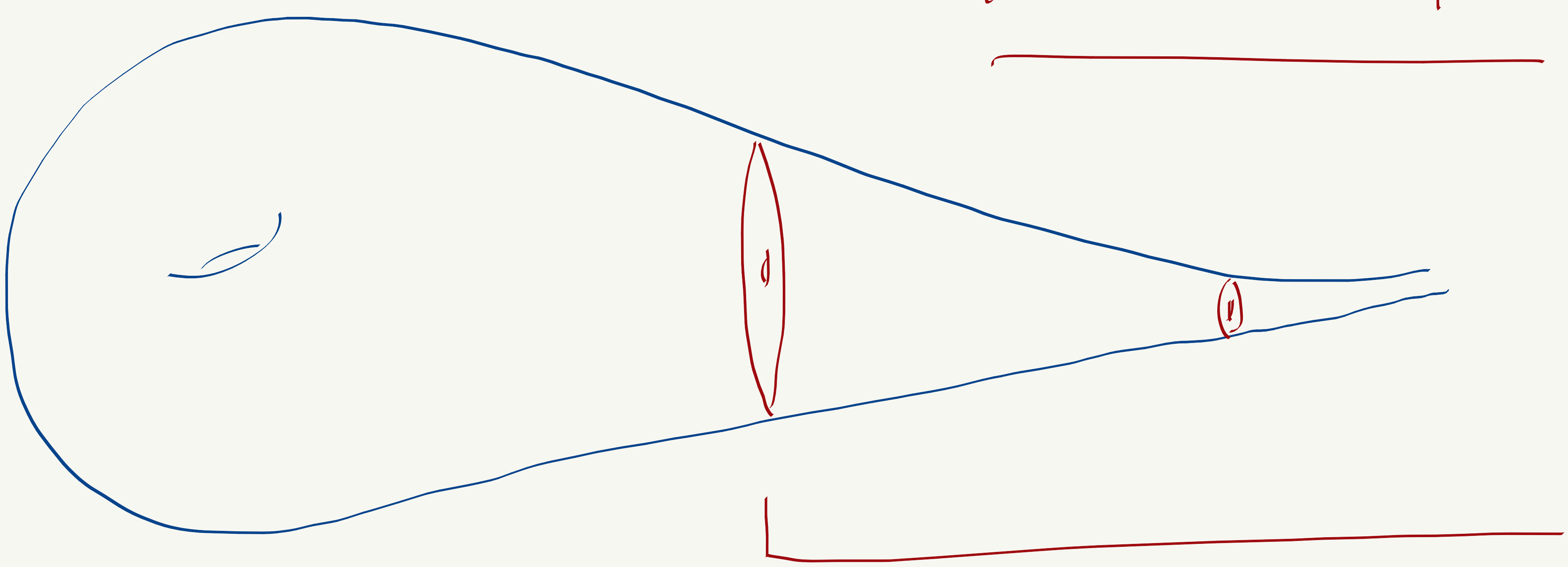
$$0 \leq \rho_{b_i}(x) \leq \begin{cases} D_1(N,i)\, e^{-(N-1-2i)R} & \text{if } N-1-2i > 0; \\ \dfrac{D_2(N,i)}{R} & \text{if } N-1-2i = 0. \end{cases}$$

Proof in a particular case: Twisted Double of a Real-Hyperbolic N-Manifold with Toral Cusps.

$K = \Gamma \backslash \mathbb{H}^N_{\mathbb{R}}$ finite volume with toral cusps.

Pictorially, we have

$$C = [0,\infty) \times T^{N-1}$$
$$g_{-1} = ds^2 + e^{-2s} g_{T^{N-1}}$$



s (height parameter)

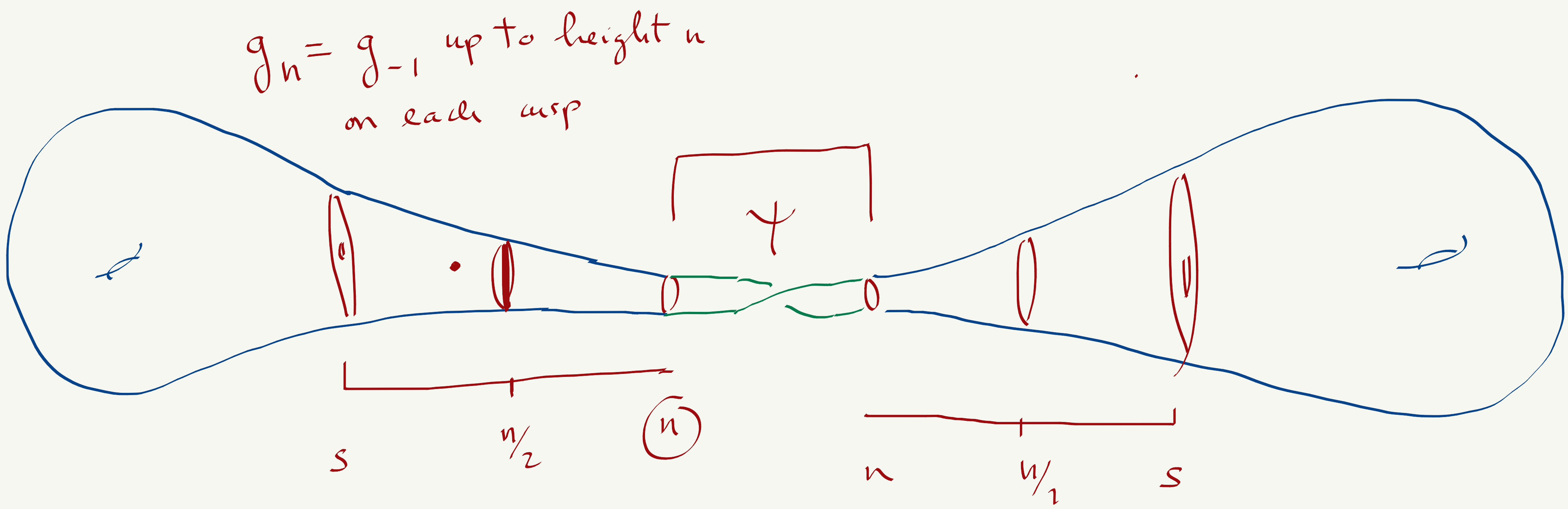
$K \Rightarrow \overline{K}$ manifold with boundary obtained by chopping the cusps.

Given two copies of $\overline{K}$ I can construct a closed manifold

$$M = \overline{K} \underset{\tau}{\#} \overline{K}$$, τ affine diffeomorphism.

Twisted double

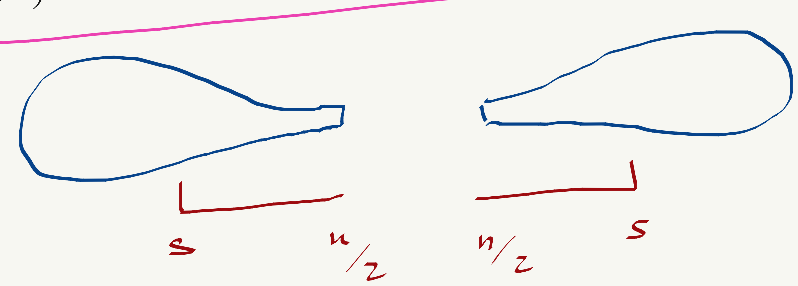
Next, I want to equip $M = \bar{K} \#_{Y} \bar{K}$ with a SEQUENCE of metrics $\{g_n\}$ having the following properties



with $|\sec_{g_n}| \le B$ (B independent of n!) and

$$\boxed{\lim_{n\to\infty} \operatorname{Vol}_{g_n}(M) = 2 \operatorname{Vol}_{g_{-1}}(K)} \quad (3)$$

Also, for any n we define the following open disconnected subset of (M, g_n)

$M_n :=$ 

and notice that

$$\boxed{\operatorname{Vol}_{g_n}(M_n) = \operatorname{Vol}_{g_n}(M) - \varepsilon(n), \quad \lim_{n\to\infty} \varepsilon(n) = 0} \quad (4)$$

$\Lambda = \pi_1(M)$ is residually finite $\Longrightarrow$ take a sequence $\{\Lambda_k\}$

and consider

Riemannian Regular Cover

$$(M^k, g^k_{n}) \longrightarrow (M, g_{n})$$

where $g^k_n := \pi^*_k g_n$. Finally, let's define

$$\boxed{M^k_n := \pi_k^{-1}(M_n) \subset M^k.}$$

By studying $(M^k, g^k_n) \longrightarrow (\tilde{M}, \tilde{g}_n)$, $\tilde{g}_n = \pi^* g_n$

where $\pi : \tilde{M} \longrightarrow M$ is the universal cover, one

can show that there exists $k_0 = k_0(n)$ such that

for any $k \geq k_0(n)$ and any $p \in M^k_n$, there

exists a ball centered at p of radius $n/2$

isometric to a ball $B_{n/2}(-)$ inside $(\mathbb{H}^N_{\mathbb{R}}, g_{-1})$.

We can now use the "Price Package" (cf. (1) and (2))

in degrees $i \leq \frac{N-1}{2}$ as follows:

For any k in the tower and for fixed n :

$$b_i(M^k) \underset{\uparrow}{=} \int_{M^k} \rho_{b_i} \, d\mu_{g_n^k} = \int_{M_n^k} \rho_{b_i} \, d\mu_{g_n^k} + \int_{M^k \setminus M_n^k} \rho_{b_i} \, d\mu_{g_n^k}$$

R

$$\leq \begin{cases} D_1(N,i)\, e^{-(N-1-2i)\frac{n}{2}} \mathrm{Vol}_{g_n^k}(M_n^k) + C(N,i)\, \mathrm{Vol}_{g_n^k}(M^k \setminus M_n^k) & \text{if } N-1-2i>0; \\ \frac{2}{n} D_2(N,i)\, \mathrm{Vol}_{g_n^k}(M_n^k) + C(N,i)\, \mathrm{Vol}_{g_n^k}(M^k \setminus M_n^k) & \text{if } N-1-2i=0, \end{cases}$$

We then have that (see (4)):

$$\boxed{\frac{b_i(M^k)}{\mathrm{Vol}_{g_n^k}(M^k)} \leq f(n) + \frac{C(N,i)\,\varepsilon(n)}{\mathrm{Vol}_{g_n}(M)}} \quad (5)$$

where

$$\boxed{\lim_{n \to \infty} f(n) = 0 \,!} \quad (6)$$

But now

$$\boxed{\mathrm{Vol}_{g_n^k}(M^k) = \deg(\pi_k)\,\mathrm{Vol}_{g_n}(M)}$$

and

$$\lim_{n\to\infty} \mathrm{Vol}_{g_n}(M) = \underline{2\,\mathrm{Vol}_{g_{-1}}(K)} \quad \text{(see (3))}$$

so that by letting $n \to \infty$ we conclude that

$$\boxed{\overline{\lim_{k\to\infty}} \frac{b_i(M^k)}{\deg(\pi_k)} = 0}$$

for $\boxed{i \in N - 1/2}$ (see (5) and (6)).

We then conclude the proof by Poincaré duality and by using Mayer-Vietoris to show that

$$\boxed{\chi(M) = 2\,\chi(K)}$$

Thanks for listening!!

It was a great pleasure to speak here, and

I hope to meet you all one day in person!