Discussion Notes 3

Sept 9, 2024

Last

L2 : More IBP

L3 : Trigonometric Integrals

Today

L4 : Trigonometric Substitution

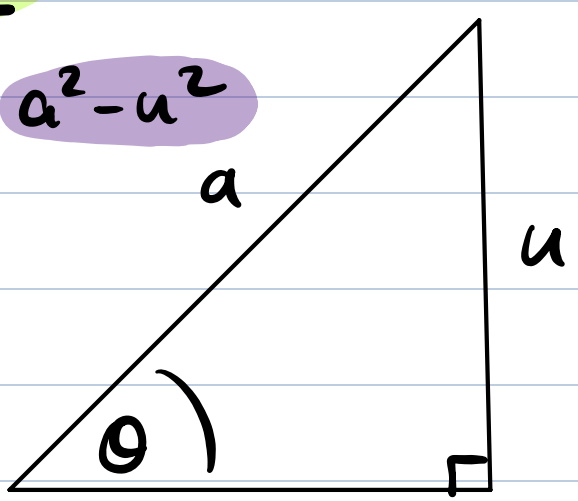
L5 : More Trigonometric substitution

* $\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\csc^2 x$

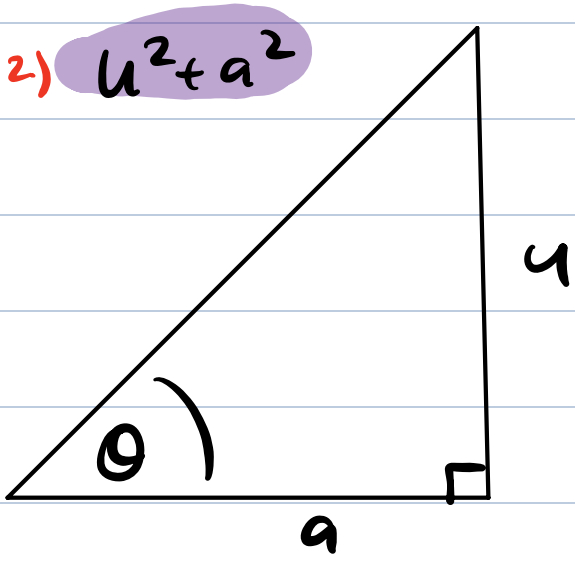
Visual

1.) $a^2 - u^2$



$$\sin(\theta) = \frac{u}{a}$$

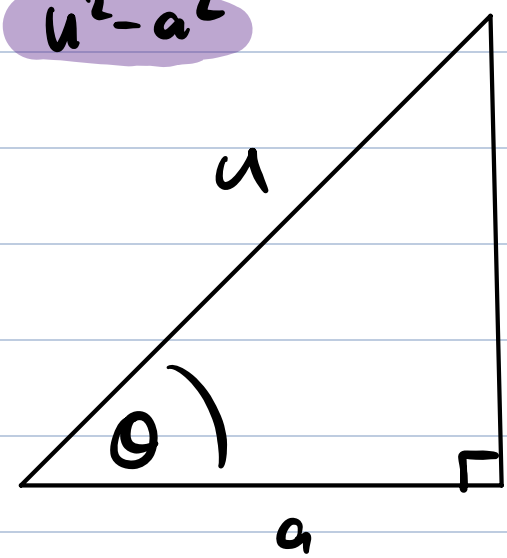
2) $u^2 + a^2$



$$\tan(\theta) = \frac{u}{a}$$

$$(\tan^2\theta + 1 = \sec^2\theta)$$

3.) $u^2 - a^2$



$$\sec(\theta) = \frac{u}{a}$$

From $\sin^2\theta + \cos^2\theta = 1$

we get

1.) $1 - \sin^2\theta = \cos^2\theta$

2.) $\tan^2\theta + 1 = \sec^2\theta$

3.) $\sec^2\theta - 1 = \tan^2\theta$

Problem 1 (Spring 2024, Exam 2, problem 6)

$$\int \frac{1}{\sqrt{-x^2-4x+5}}\,dx$$

well $-x^2-4x+5 = (x^2+4x-5)$ where

$(x+2)^2 = x^2+4x+4$

So $(x+2)^2 - 9 = x^2+4x+4-9 =$

want x^2+4x-5.

Then $9-(x+2)^2 = -(x^2+4x-5)$

So $\int \frac{1}{\sqrt{9-(x+2)^2}}\,dx$, $x+2 = 3\sin(\theta)$

$dx = 3\cos(\theta)\,d\theta$

$9 = 3^2$

$= \int \frac{3\cos\theta\,d\theta}{\sqrt{9-(3\sin\theta)^2}}\,d\theta$

$\sqrt{9-9\sin^2\theta}$

$= \sqrt{9(1-\sin^2\theta}$

$= 3\sqrt{1-\sin^2\theta}$

$= 3\sqrt{\cos^2\theta}$

$= 3\cos(\theta)$,

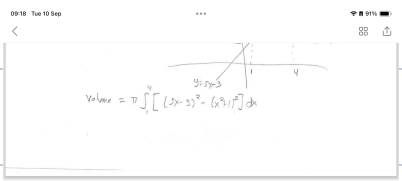
$0 \le \theta < \frac{\pi}{2}$

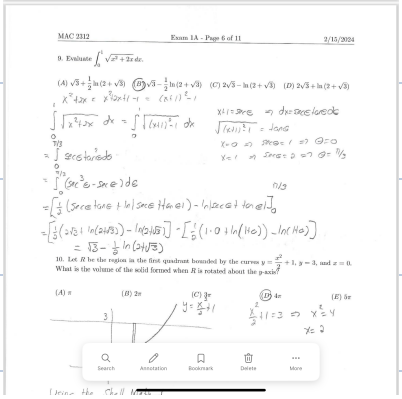
$= \frac{3}{3}\int \frac{\cos\theta\,d\theta}{\cos\theta}$

$= \int d\theta = \theta = \arcsin\left(\frac{x+2}{3}\right) + C$

Problem (Sp. 24, Exam 2, Pr 9)

$$\int_0^1 \sqrt{x^2+2x}\,dx$$





$$(x+1)^2-1 = x^2+2x+1-1$$

$$\int_0^1 \sqrt{(x+1)^2-1}\,dx$$

var − const

$x+1 = \sec\theta$, $dx = \sec\theta\cdot\tan\theta\,d\theta$

$$\int \sqrt{\sec^2\theta-1}\cdot\sec\theta\cdot\tan\theta\,d\theta$$

$$= \int \sqrt{\tan^2\theta}\cdot\sec\theta\cdot\tan\theta\,d\theta$$

$$= \int \tan^2\theta\cdot\sec\theta\,d\theta$$

$$= \int(\sec^2\theta-1)\cdot\sec\theta\,d\theta$$

$$= \int \sec^3\theta\,d\theta - \int\sec\theta\,d\theta$$

$$= \frac{1}{2}(\sec\theta\cdot\tan\theta+\ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta| + C$$

And fill in the bounds.

Problem 3: (Fall 23 , Ex 1, nr a)

$$\int \frac{x}{\sqrt{x^2+x+2}}\, dx$$

Problem 4 :

$$\int \sqrt{x^2-1}\, dx$$

$\sin^2 + \cos^2 = 1$

$\tan^2 + 1 = \sec^2$

$\sec^2 - 1 = \tan^2$

Choose $x = \sec(\theta)$, $dx = \sec(\theta)\tan(\theta)$

Then

$$\int \sqrt{\sec(\theta)^2-1} \cdot \sec(\theta)\cdot\tan(\theta)\, d\theta$$

$$= \int \tan\theta \cdot \sec\theta \cdot \tan\theta \, d\theta$$

$$= \int \tan^2\theta \cdot \sec\theta \, d\theta$$

$$= \int (\sec^2\theta - 1)\sec(\theta)\, d\theta$$

$$= \int \sec^3\theta \, d\theta - \int \sec\theta \, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^3\theta}\, d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos^3\theta}\, d\theta$$

$$= \int \frac{1}{\cos^3\theta}\, d\theta - \int \frac{1}{\cos\theta}\, d\theta$$

$$= \int \sec^3\theta \, d\theta - \int \sec\theta \, d\theta$$

$$= \underbrace{\qquad\qquad} - \ln|\sec\theta + \tan\theta| + C$$

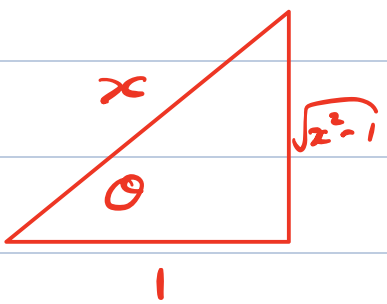
? see below

$$= \frac{1}{2}(\sec\theta\cdot\tan\theta + \ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{2}\sec\theta\cdot\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.$$

$$= \frac{1}{2}\cdot x\cdot\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| + C$$

$\frac{x}{1} = \sec\theta$



$u = \sec\theta$, $v = \tan(\theta)$

$du = \sec\theta\tan\theta$, $dv = \sec^2(\theta)\,d\theta$

So

$$\int \sec^3\theta\,d\theta = \sec\theta\cdot\tan(\theta) - \int \tan\theta\cdot\sec\theta\cdot\tan\theta$$

$$\int \sec^3(\theta)\,d\theta = \sec\theta\cdot\tan\theta - \int(\sec^2(\theta)-1)\sec(\theta)$$

$$\int \sec^3\theta\,d\theta = \sec\theta\cdot\tan\theta - \int \sec^3\theta + \int \sec\theta$$

$$2\int \sec^3\theta = \sec\theta\cdot\tan\theta + \ln|\sec\theta+\tan\theta|.$$

$$\int \sec^3\theta = \frac{1}{2}\left(\sec\theta\cdot\tan\theta + \ln|\sec\theta+\tan\theta|.\right)$$

Problem 9: $\int_{-1}^{7} \sqrt{7 + 6x - x^2}\, dx$

$$(x-3)^2 = x^2 - 6x + 9$$

$$= x^2 - 6x - 7 + 16$$

$$(x-3)^2 - 16 = x^2 - 6x - 7$$

$$16 - (x-3)^2 = -x^2 + 6x + 7$$