Discussion Class

L24, L25

Nov 9, 2023

Last Time:

- The Mean Value Theorem
- The 1^{st} Derivative Test
- Concavity and 2^{nd} Derivative Test

Today:

- L'Hospital's Rule
- Curve Sketching

1.) L'Hospital's Rule

<u>Thm</u>: Suppose f, g are differentiable and $g'(x) \neq 0$ near a (allowed to be 0 at a).

If • $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a}$

or

•

1) • Indeterminate Form : "$\frac{0}{0}$", "$\pm\frac{\infty}{\infty}$"

2) • Indeterminant Product : "$0 \cdot \infty$"

3) • Indeterminant Difference : "$\infty - \infty$"

4) • Indeterminant Powers : "0^0", "∞^0", "1^∞"

- we can only apply L'Hospital to limits of indeterminant form, but limits of type 2, 3, and 4 can be changed to limits of type 1.

Example 1: Evaluate $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ " $\infty-\infty$ "

We want to use L'Hospital's rule

$$\lim_{x\to 0}\frac{1}{x}-\frac{1}{\sin x}=\lim_{x\to 0}\frac{\sin x - x}{x\cdot\sin x} \quad "\frac{0}{0}"$$

$$\overset{L'H}{=}\lim_{x\to 0}\frac{\cos(x)-1}{\sin x + x\cdot\cos x} \quad "\frac{0}{0}"$$

$$\overset{L'H}{=}\lim_{x\to 0}\frac{-\sin(x)}{\cos x + \cos x + x\cdot(-\sin x)}$$

$$=\frac{0}{1+1+0}$$

$$=0.$$

Example 2: Evaluate $\lim_{x\to\infty} x^{\ln x}$ " ∞^{∞} "

Not an indeterminant form

We know $\lim_{x\to\infty} x^{\ln x}=\lim_{x\to\infty} e^{\ln(x^{\ln x})}$

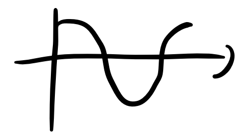
$$=\lim_{x\to\infty} e^{\ln(x)\cdot\ln(x)}$$

$$=e^{\lim_{x\to\infty}\ln(x)^2}$$

Where $\lim_{x\to\infty}\ln(x)^2=\left(\lim_{x\to\infty}\ln(x)\right)^2=\infty$

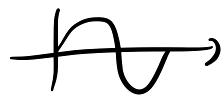
So $\lim_{x\to\infty} x^{\ln(x)}=\infty$

Example 3: $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\sin(\cos(x))}{\cos(x)}$ "$\frac{0}{0}$"



$\cos(\frac{\pi}{2}) = 0$

$\sin(0) = 0$



$$\overset{L'H}{=} \lim_{x \to \frac{\pi}{2}} \frac{\cos(\cos(x)) \cdot (\cancel{-\sin(x)})}{\cancel{-\sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \cos(\cos(x))$$

$$= \cos(\cos(\lim_{x \to \frac{\pi}{2}} x))$$

$$= \cos(\cos(\tfrac{\pi}{2}))$$

$$= \cos(0)$$

$$= 1$$

2.) Curve Sketching

Information we need to sketch:

- Domain.
- Vertical and horizontal asymptotes.
- Intervals of increasing and decreasing.
- Extreme Values.
- Intervals of concavity.
- Intercepts.

Example 1 : $f(x) = \ln(x^2+1)$

- Domain : $(-\infty, \infty)$
- Horizontal asymptotes

$$\left.\begin{aligned} \lim_{x\to\infty} \ln(x^2+1) &= \infty \\ \lim_{x\to-\infty} \ln(x^2+1) &= -\infty \end{aligned}\right\} \text{No Horizontal Asymptotes}$$

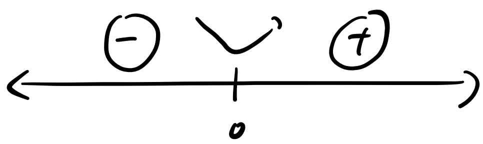
- Vertical asymptotes

$\lim_{x\to a} \ln(x^2+1) = \ln(a^2+1)$ since f continuous on $(-\infty, \infty)$

– No vertical asymptotes

- – Increasing : $f'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$

$(0, \infty)$

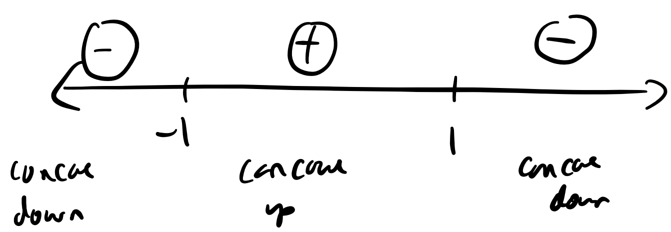


- – Decreasing

$(-\infty, 0)$

- Local minimum at 0.
- Concavity : $f''(x) = \frac{2(x^2+1) - (2x)(2x)}{(x^2+1)^2}$

$$= \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$



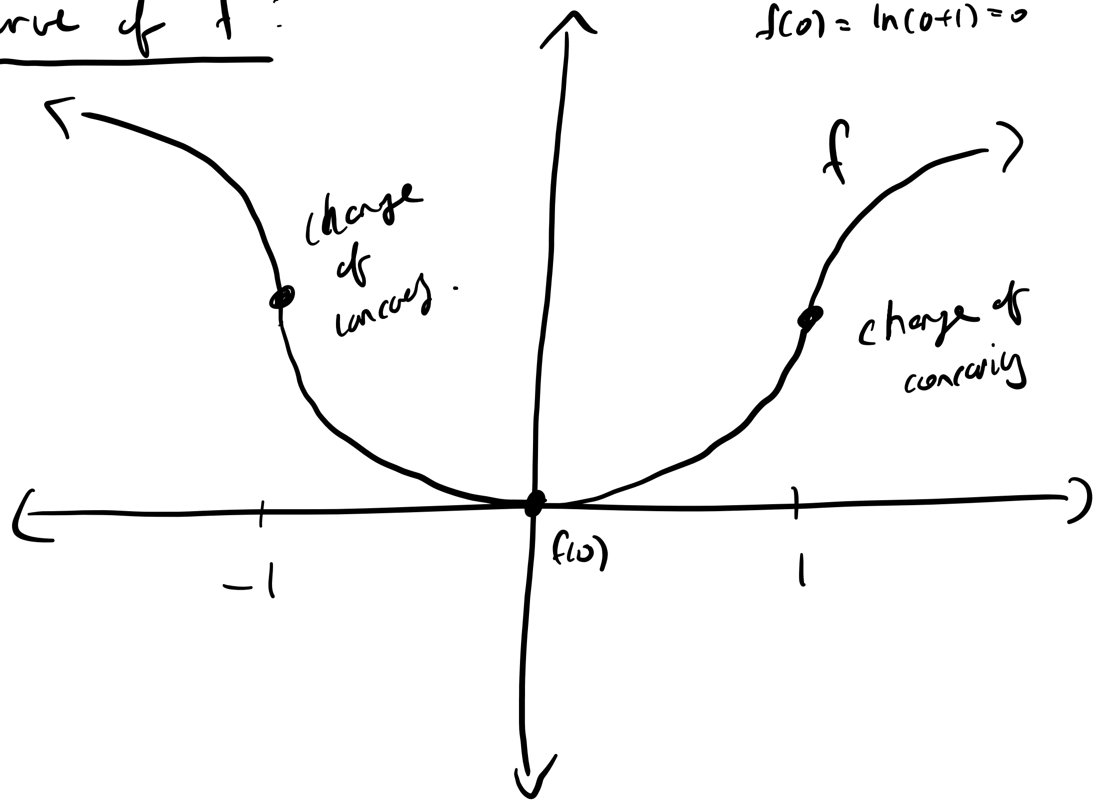
setting equal to 0

$2-2x^2 = 0$

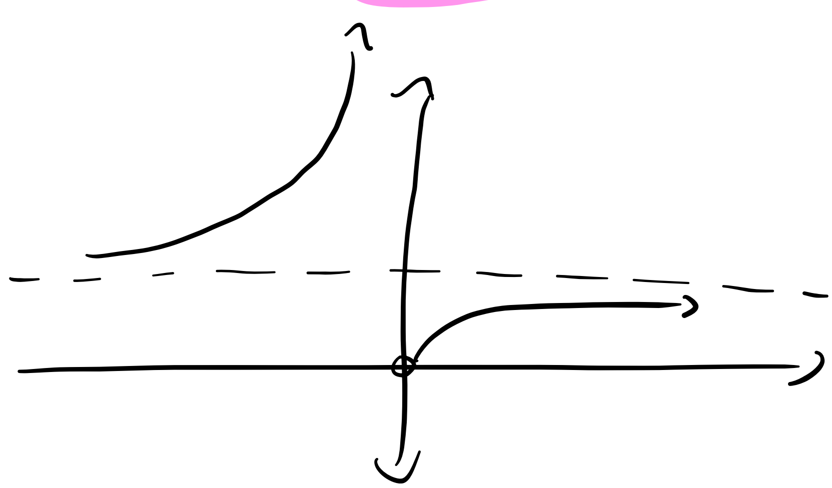
$\Rightarrow x = \pm 1$

Curve of f:

$f(0) = \ln(0+1) = 0$



Example 2 $f(x) = e^{-\frac{1}{x}}$



Example 3:

$$f(\theta) = 2\cos(\theta) + \cos^2\theta \quad , \quad [0, 2\theta]$$

Example 4: $f(x) = x\sqrt{6-x}$