# Anti-Derivatives and Riemann Sums

April 9, 2024

## Last time

- Applied Optimization

## Today

- Riemann Sums.
- Anti-derivatives.

**def:** An **anti-derivative** of a function $f(x)$ is a function $F$ s.t. $F'(x) = f(x)$.

**def:** (Riemann Sums)
let $f : [a,b] \to \mathbb{R}$.

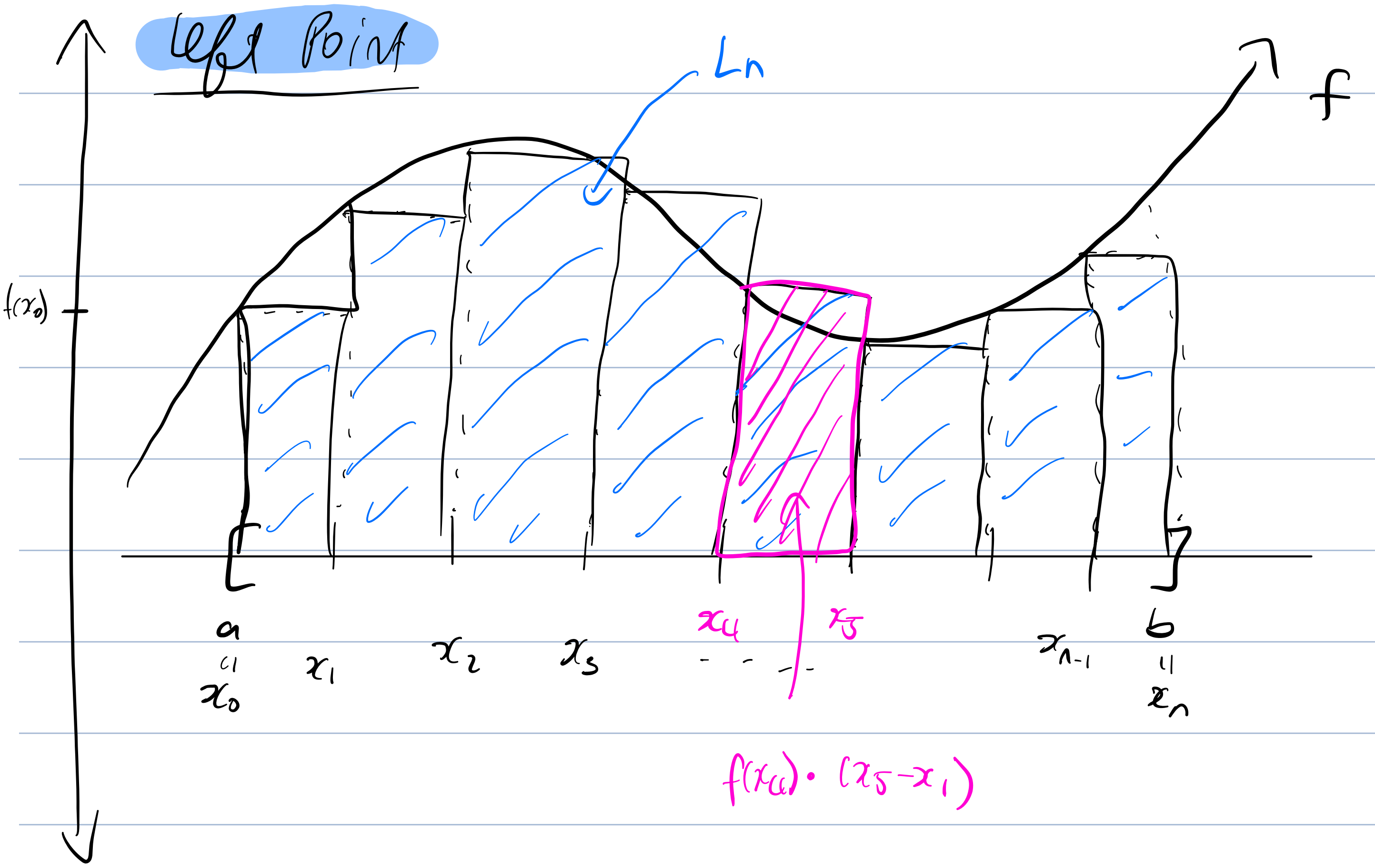
Left Endpoint : $L_n := \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_0)$

Right Endpoint : $R_n := \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$

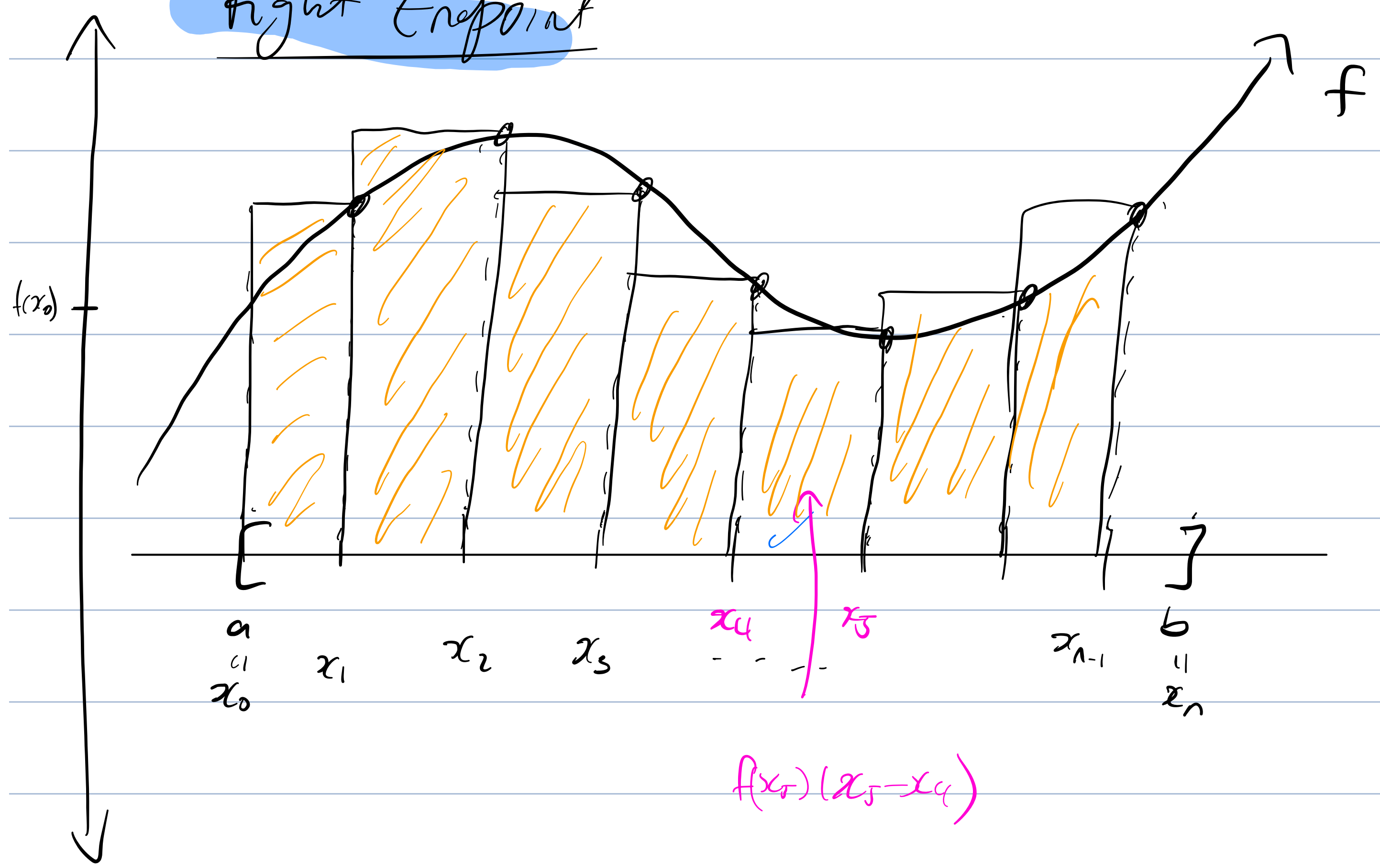
Midpoint : $M_n := \sum_{i=1}^{n-1} f(x_i^*)(x_i - x_{i-1})$

summing up areas of rectangles

## Left Point

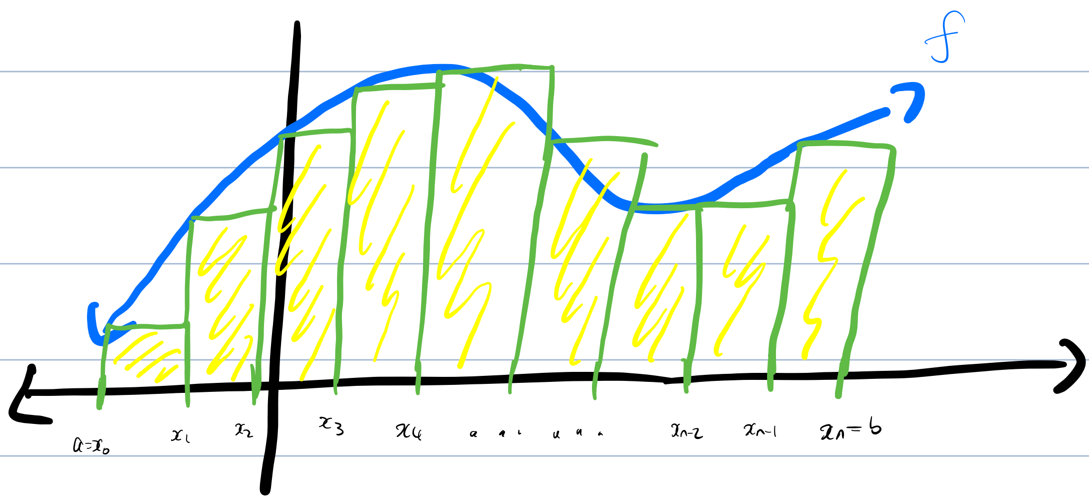



## Right Endpoint



# Areas and Riemann Sums:

**The Area Problem:** Given a curve, how do we find the area under the curve?




- **Riemann Sum**: Given $f:[a,b]\to\mathbb{R}$, $a=x_0<x_1<x_2<\dots<x_n=b$

the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ where $x_i^*\in[x_{i-1},x_i]$ is called a Riemann sum.

- **Right, Middle, left-endpoint approximations:** (Examples of Riemann Sums)

, $[x_{i-1},x_i]$

- $R_n=\sum_{i=1}^{n} f(x_i^*)\Delta x$ , $x_i^*=x_i$ (right end point approximation)
- $L_n=\sum_{i=1}^{n} f(x_i^*)\Delta x$ , $x_i^*=x_{i-1}$ (left end point approximation)
- $M_n=\sum_{i=1}^{n} f(x_i^*)\Delta x$ , $x_i^*=x_{i-1}+\frac{\Delta x}{2}$ (middle point approximation)

- **Definite Integral:**

Given a function $f:[a,b]\to\mathbb{R}$ we define

$$\int_a^b f(x)\,dx := \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $a=x_0<x_1<x_2<\dots<x_n=b$ a partision of $[a,b]$ and $x_i^*\in[x_{i-1},x_i]$

If the limit exists, then we say $f$ is Riemann Integrable

- There are functions that are not Riemann integrable.

But this need not be a problem since all the functions in this course are continuous, or continuous except at finitely many points:

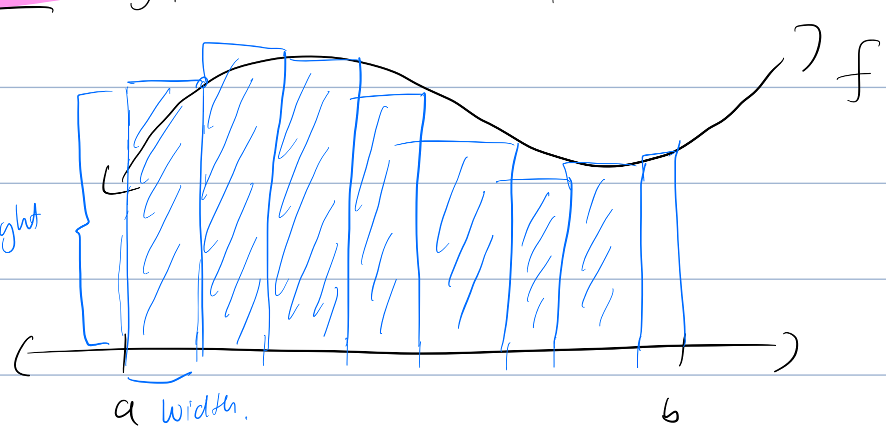
(Special Case: Lebesgue's Criterion for Riemann Integrability)

**Thm:** If $f: [a,b] \to \mathbb{R}$ continuous, or if $f$ has at most finitely many points of discontinuity, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$ exists.

## 3 ways to visualize Integrals

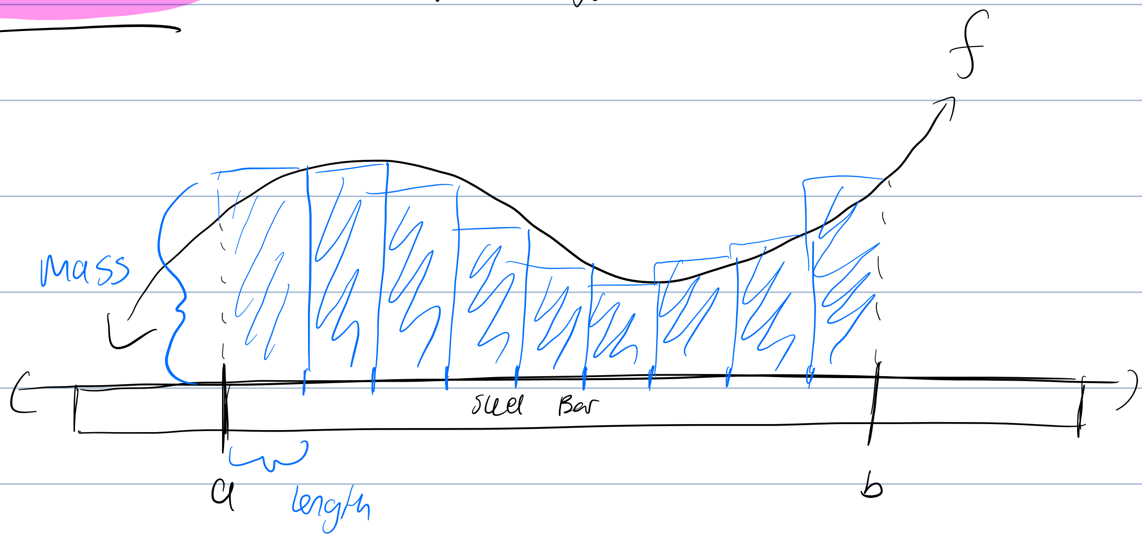
Area, Mass, Electric charge

**1. Area:** $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$, $a = x_0 < x_1 < \dots < x_n = b$.



## 2. Mass :

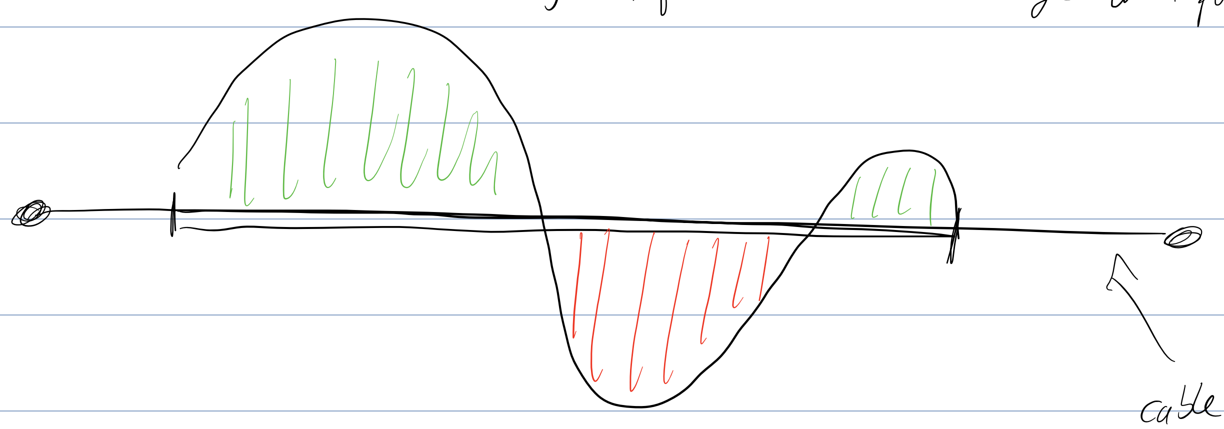
$$\text{density} = \frac{\text{mass}}{\text{volume}} \Rightarrow \text{mass} = \text{density} \cdot \text{volume}$$




$f$ : input is point on the rod and output is density at that point.

## 3.) Net Electric Charge

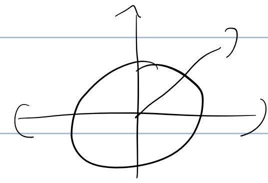
$f$ - represents electric charge at a point

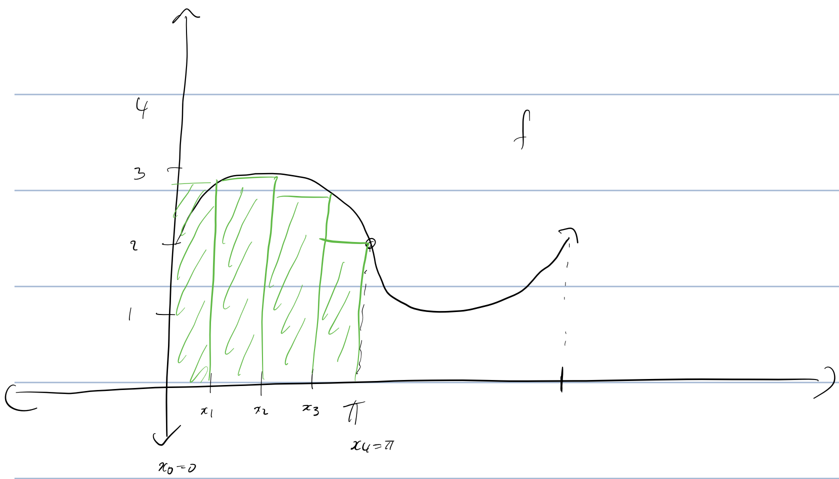



$\int_a^b f(x)\,dx$ = Net charge on the rod from $[a,b]$

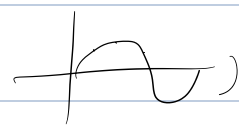
**Example:** Evaluate the upper and lower sum for $f(x) = 2+\sin x$

$n=4$, $0$, $\pi$






$$R_4 = \sum_{i=1}^{4} f(x_i)\cdot\Delta x \quad , \quad \Delta x = \frac{\pi-0}{4} = \frac{\pi}{4}$$



$x_1 = 0+\frac{\pi}{4} = \frac{\pi}{4}$ , $f(\frac{\pi}{4}) = 2+\frac{\sqrt{2}}{2}$

$x_2 = 0+2\cdot\frac{\pi}{4} = \frac{\pi}{2}$ , $f(\frac{\pi}{2}) = 2+1$

$x_3 = 0+3\cdot\frac{\pi}{4} = \frac{3\pi}{4}$ , $f(\frac{3\pi}{4}) = 2+\frac{\sqrt{2}}{2}$

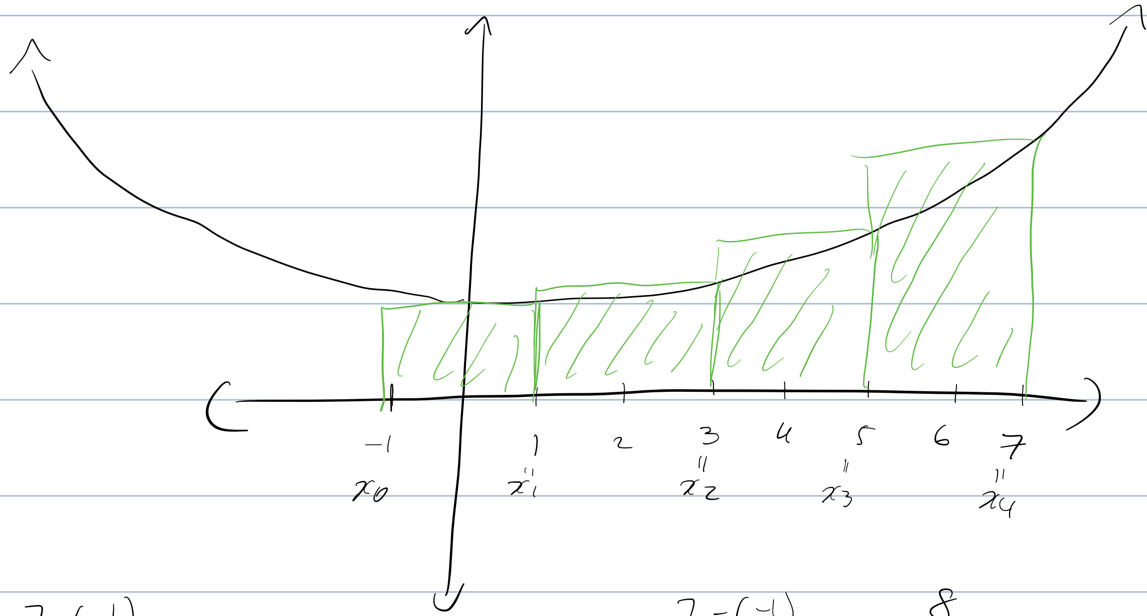
$x_4 = 0+4\cdot\frac{\pi}{4} = \pi$ , $f(\pi) = 2+0$

Then $R_4 = \sum_{i=1}^{4} f(i\cdot\frac{\pi}{4})\cdot\frac{\pi}{4} = (9+\sqrt{2})\cdot\frac{\pi}{4}$

$$L_4 = \sum_{i=0}^{3} f(x_i)\cdot\Delta x = (9+\sqrt{2})\cdot\frac{\pi}{4}$$

**Example** ( Final Exam Fall 2022)

Given $f(x) = x^2+1$ on $[-1,7]$. Find the right endpoint Riemann sum approximation of $\int_{-1}^{7} f(x)\,dx$ with $n=4$ rectangles.




$\Delta x = \frac{7-(-1)}{4} = 2$ , $\Delta x = \frac{7-(-1)}{n} = \frac{8}{n}$

And $x_i = x_0 + i \cdot \Delta x = -1 + i \cdot \frac{8}{n}$

general

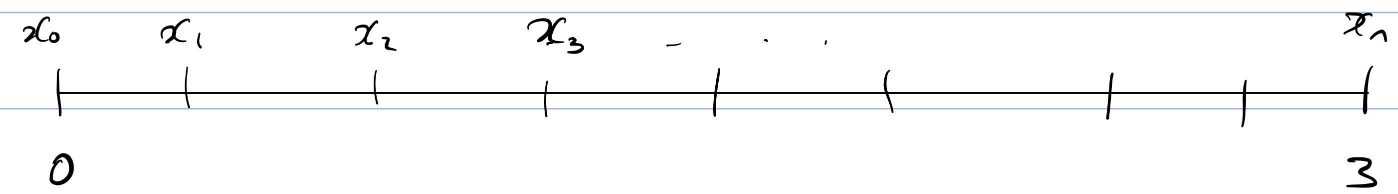
Then $$\boxed{\begin{aligned}\int_{-1}^{7} f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(-1+\frac{8i}{n}\right)^2+1\right)\cdot \frac{8}{n}\end{aligned}}$$

$n=4$

So $\boxed{R_4 = \sum_{i=1}^{4} \left((-1+2i)^2+1\right)\cdot 2}$ (Approximation)

**Example 1** : Given $f(x) = x^2 + 1$ on $[0, 3]$.

Use $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(n+2)}{6}$ to find the exact area under the curve of $f$ over $[0,3]$




- $\Delta x = \dfrac{b-a}{n} = \dfrac{3}{n}$
- $x_0 = 0, \quad x_1 = \dfrac{3}{n}, \quad x_2 = \dfrac{6}{n}, \quad \ldots, \quad x_n = 3$
- $x_i = x_0 + i \cdot \Delta x = i \cdot \dfrac{3}{n}$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left( \left(\frac{3i}{n}\right)^2 + 1 \right) \cdot \frac{3}{n}$$

$$= \left( \sum_{i=1}^{n} \frac{9i^2}{n^2} + 1 \right) \cdot \frac{3}{n}$$

$$= \left( \left( \frac{9}{n^2} \sum_{i=1}^{n} i^2 \right) + n \right) \frac{3}{n}$$

$$= \frac{27}{n^3} \sum_{i=1}^{n} i^2 + 3$$

$$= \frac{27}{n^3} \cdot \frac{n \cdot (n+1)(2n+1)}{6} + 3$$

Then $\displaystyle\int_0^3 f\,dx = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{27}{n^3} \cdot \frac{n \cdot (n+1)(2n+1)}{6} + 3$

$$= \frac{27 \cdot 2}{6} + 3 = \boxed{12.}$$

**Example 2:** Given $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$, $[0,1]$

Find a function $f$ for which the limit represents the area of under the graph. (Right endpt.)

$$\int_0^1 f\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x \quad , \quad \Delta x = \frac{1}{n}$$

$$x_i = i\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} \qquad f(x) = x^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2\cdot\frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$$

**Example 3:** Given $\int_0^3 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{9i^2}{n^2}\right)\frac{6}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \left(i\cdot\frac{3}{n}\right)^2\right)\frac{3}{n}\cdot 2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 + 2\left(i\cdot\frac{3}{n}\right)\right)^2\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{3-0}{n} = \frac{3}{n}$, $\quad x_i = x_0 + i\cdot\Delta x$

$$= i\cdot\frac{3}{n}$$

So $f(x) = 2x^2+4$.

**Example:** Find $f(x)$ given $f'(x) = 1 + 3\sqrt{x}$ and the initial condition $f(4) = 25$.

Solution: The anti-derivative of $f'(x) = 1 + 3x^{\frac{1}{2}}$

is
$$f(x) = x + \frac{3x^{\frac{3}{2}}}{3/2} + C$$
$$= x + 2x^{3/2} + C$$

Then $25 = f(4) = 4 + 2(4)^{3/2} + C$, solve $C$

$$\Rightarrow 21 = 2(\sqrt{4})^3 + C$$
$$\Rightarrow 21 = 2 \cdot 2^3 + C$$
$$\Rightarrow C = \frac{21}{16}$$

Thus the solution to the above equation is

$$\boxed{f(x) = x + 2x^{3/2} + \frac{21}{16}}$$

# Example :

Find the anti-derivatives of the following function. Use C to denote the constant.

Anti - Derivative

1. $$(x^2-1)^2 = x^4 - 2x^2 + 1 \rightsquigarrow \frac{x^5}{5} - \frac{2x^3}{3} + x + C$$

2. $$\sqrt{\frac{3}{z}} = \sqrt{3}\, z^{-\frac{1}{2}} \rightsquigarrow \frac{\sqrt{3}\, z^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{3z} + C$$

3. $$\frac{4+u^2}{u} = \frac{4}{u} + u \rightsquigarrow 4\ln|u| + \frac{u^2}{2} + C$$

4. $$cos(\theta) \rightsquigarrow \sin(\theta)$$

5. $$-2cos(\theta)sin(\theta) = -\sin(2\theta) \rightsquigarrow \frac{\cos(2\theta)}{2} + C$$

6. $$cos(\theta)sin(\theta) = \frac{1}{2}(2\sin\theta\cos\theta) = \frac{1}{2}(\sin(2\theta)) = \frac{1}{2}\left(\frac{-\cos(2\theta)}{2}\right)$$

$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

7. $$\frac{4}{\sqrt{1-x^2}} \rightsquigarrow 4\arcsin(x) + C$$

8. $$e^{2u+1} \rightsquigarrow \frac{e^{2u+1}}{2} + C$$

9. $$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \rightsquigarrow \begin{cases} \frac{x^2}{2} + C_1, & x \geq 0 \\ -\frac{x^2}{2}, & x < 0 \end{cases}$$