Discussion Note 12

April 10, 2024

Last Time:

- Riemann Sums
- Anti-derivatives

Today

- The definite integral
- The FTC (next time)

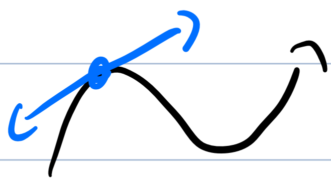
High-level Overview

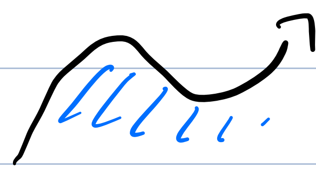
① limits

② Derivatives

③ Integrals

FTC





Operations In Calculus :

(Limits, Derivatives, Integrals)

	Input	Output
Limits	function f number a	Number (provided exists) $\lim_{x \to a} f(x)$
Derivatives	function f	function f'
Integrals (Definite)	function f, interval $[a,b]$	Number $\int_a^b f(x)\,dx$,

These Basic properties make calculating these objects easier.

Definition (The Definite Integral)

Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function.
Then **definite integral** of f over $[a,b]$ is

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x_n}_{M_n,\ R_n,\ L_n}$$

Remark: The limit on the right hand side always exists for integrable f.

Properties of the Definite Integral

Let $f:[a,b] \to \mathbb{R}$ be continuous.

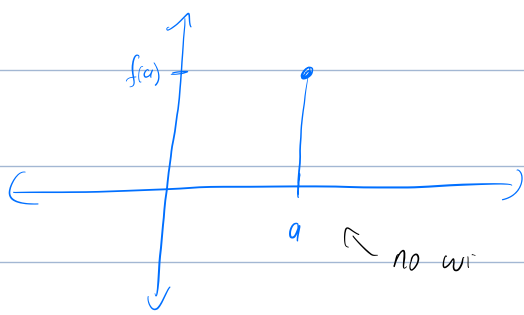
How is a definite Integral defined?

Through sums and limits. Therefore if a property is preserved by sums and limits it preserved by definite integrals.

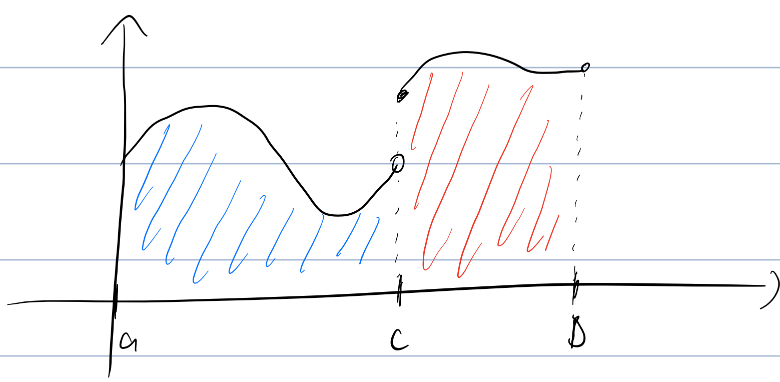
1) Constant multiple: $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ (special case $\int_a^b c\,dx = c(b-a)$)

2) Sums and differences: $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3) Endpoints equal: $\int_a^a f(x)\,dx = 0$



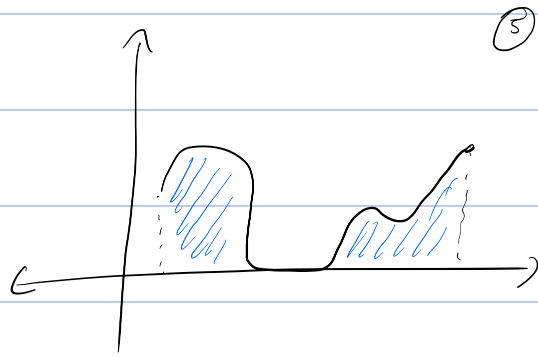
4) Splitting: For $a \leq c \leq b$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



More properties

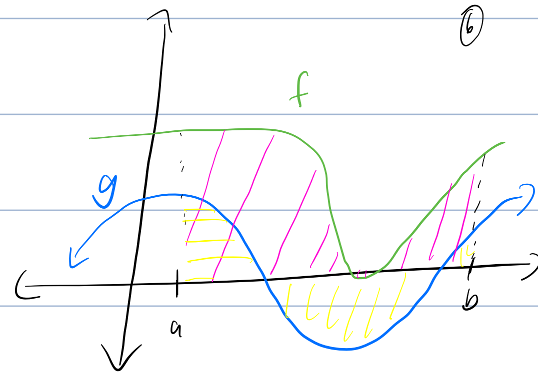
5.) If $f(x) \geq 0$ for all x in $[a,b]$,

then $\int_a^b f(x)\,dx \geq 0$.



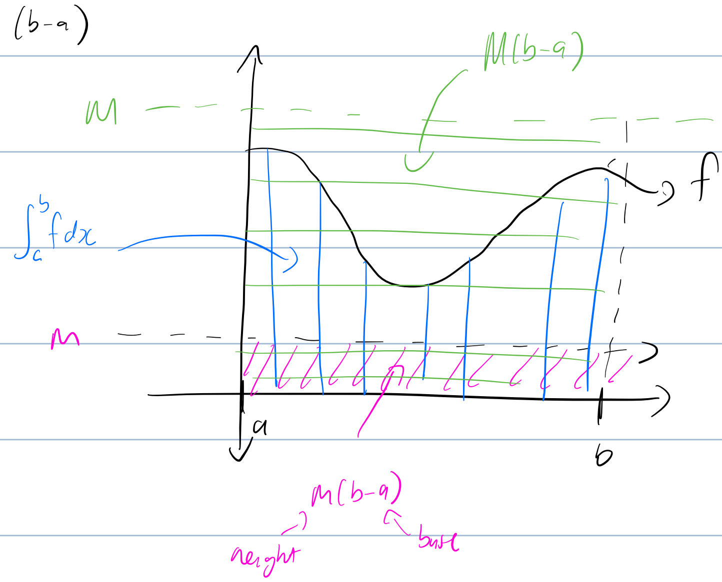
6.) If $g(x) \leq f(x)$ for every x in $[a,b]$,

then $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$.



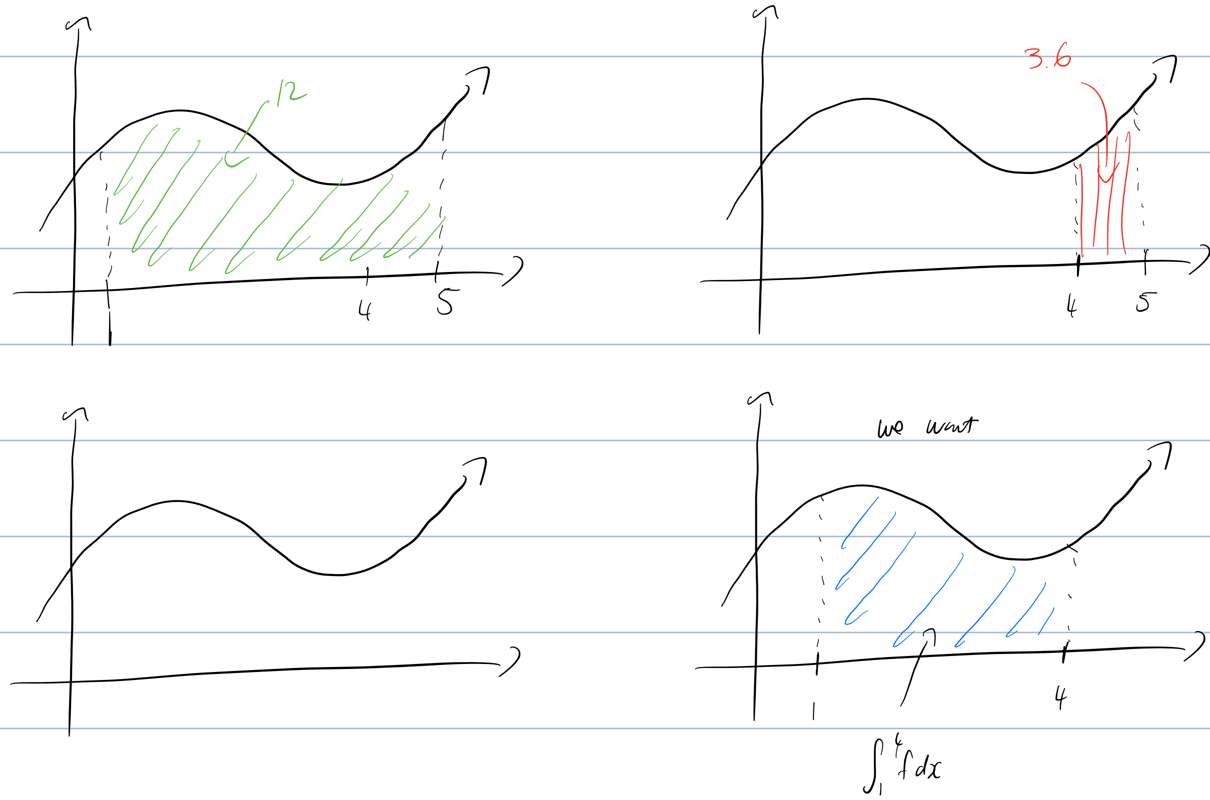
7.) If $m \leq f(x) \leq M$ for all x in $[a,b]$,

then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$



Examples: Use properties of the definite integral to find the following:

Example 1: Given $\int_1^5 f\,dx = 12$, $\int_4^5 f\,dx = 3.6$, find $\int_1^4 f\,dx$



$$\text{So} \quad \int_1^5 f\,dx - \int_4^5 f\,dx = \int_1^4 f\,dx$$

$$\Rightarrow \int_1^4 f\,dx = 12 - 3.6 = \underline{\underline{8.4}}$$

Example 2:

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1}\,dx = 0$$

Why?

Example 3:

$$\int_{6}^{-10} f(x)\,dx = 23, \quad \int_{-10}^{6} g(x) = -9$$

Find

$$\int_{-10}^{6} 2f(x) - 10\,g(x)\,dx$$

$$= 2\int_{-10}^{6} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$

$$= -2\int_{6}^{-10} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$

$$= -2(23) - 10(-9)$$

$$= -46 + 90$$

$$= 44$$

Examples : Suppose f, g continuous functions.

Given $\int_{-1}^{1} f(x)\,dx = 7$, $\int_{-1}^{4} 2g(x)\,dx = 4$, $\int_{-1}^{4} [f(x) + g(x)]\,dx = 9$

What is $\int_{1}^{4} f(x)\,dx$?

- $\int_{-1}^{4} 2g(x)\,dx = 4 \Rightarrow \int_{-1}^{4} g(x) = 2$.

- $\int_{-1}^{4} f(x) + g(x)\,dx = \int_{-1}^{4} f(x)\,dx + \int_{-1}^{4} g(x)\,dx$

$\Rightarrow \quad 9 = \int_{-1}^{4} f(x)\,dx + 2$

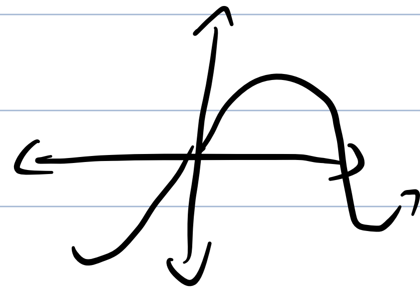
So $\int_{-1}^{4} f(x)\,dx = 7$

- Then $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$

$\Rightarrow \quad 7 = 7 + \int_{1}^{4} f(x)\,dx$

$\Rightarrow \quad \boxed{\int_{1}^{4} f(x)\,dx = 0.}$

Example 1 : $\frac{1}{2\pi}\int_{-1}^{1} \cos(n\theta)\, d\theta$



$$= \frac{1}{2\pi} \left. \frac{\sin(n\theta)}{n} \right|_{-1}^{1}$$

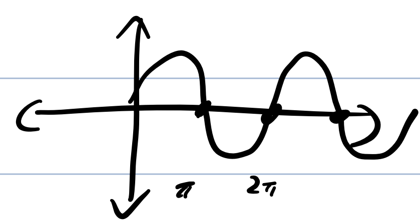
$$= \frac{1}{2\pi n}\Big(\sin(n) - \sin(-n)\Big)$$

$$= \frac{1}{2\pi n}\, 2\sin(n)$$

$$= \frac{\sin(n)}{\pi n}.$$

Example 2 : $\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\theta)\, d\theta$, $n \neq 0$

$$= \frac{1}{2\pi}\left. \left(\frac{\sin(n\theta)}{n}\right)\right|_{-\pi}^{\pi}$$



$$= \frac{1}{2\pi n}\Big(\sin(n\pi) - \sin(-n\pi)\Big)$$

$$= \frac{1}{2\pi n}(0 - 0)$$

$$= 0$$

Example: $\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, d\theta \quad , \quad n \neq 0$

$= 0$, because $\sin(-x) = -\sin(x)$ and we are integrating about an interval centered about 0.