Discussion Class 2

January 14, 2024

Last Time:

- Introductions
- Solving equations

Today (L1 - L4):

- Absolute Value
- Functions (Domain, Range, Construction, inverses, logarithms)

Functions:

- The main mathematical object we will work with in this class is functions

1.) We study limiting behavour of functions at specific points.

2.) The definition of a derivative is dependend on functions and limits.

Functions : A rule / map where every point in the domain gets sent to a unique point in the codomain.

1. Typos of functions

- Polynomials : $-10x^3 + 5x + 3$
- Power functions : $\sqrt{x}$, $x^{\frac{1}{5}}$
- Trigonometric functions : $\cos(x)$, $\sin(x)$, $\cot(x)$
- Exponential functions : a^x , $a > 0$, $a \neq 1$.
- Logarithmic : $\log_b(x)$, $\ln(x)$
- Absolute value : $|x| := \begin{cases} x, & \text{if } x \geq 0. \\ -x, & \text{if } -x < 0. \end{cases}$

Functions :

2. Concepts that are included with functions.

- **Domain** : The set of input values.
- **Co. domain** : The set where the outputs live.
- **Range** : All the points in the codomain that came from a point in the domain.

Example : $f(x) = x^2$.

We can talk about f as a function from $\mathbb{R}$ (set of real numbers) to $\mathbb{R}$.

Then
- domain of f : $\mathbb{R}$
- co-domain of f : $\mathbb{R}$
- Range of f : $[0, +\infty) \neq \mathbb{R}$

- Then $\text{Range}(f) \neq \mathbb{R}$ because we cannot find a negative number $y<0$ in the co-domain and number x in domain s.t $f(x) = x^2 = y$.

Functions:

3. New Functions from old.

Mathmaticians like creating new Mathematical objects and studying them.

a. Algebraic operations:

Example: $f(x) = \sin(x)$ and $g(x) = \sqrt{x+1}$

- Then $f+g$ is a new function where

$$(f+g)(x) = \sin(x) + \sqrt{x+1}$$

- $f/g\,(x) = \dfrac{\sin(x)}{\sqrt{x+1}}$, $fg(x) = \sin(x) \cdot \sqrt{x+1}$

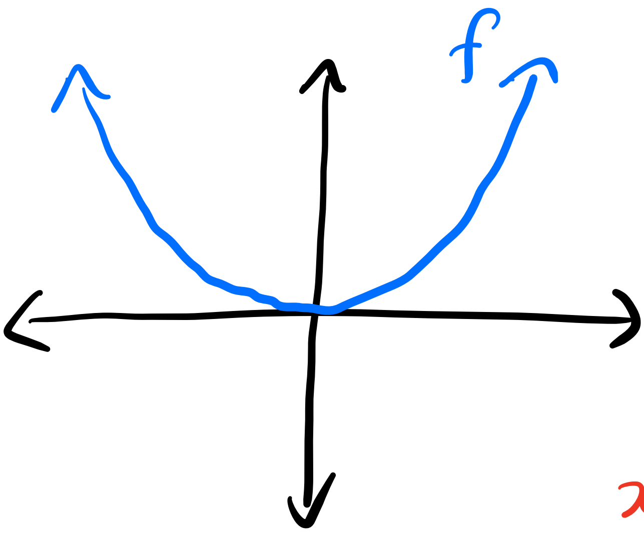
Warning: These new functions have new domains and ranges.

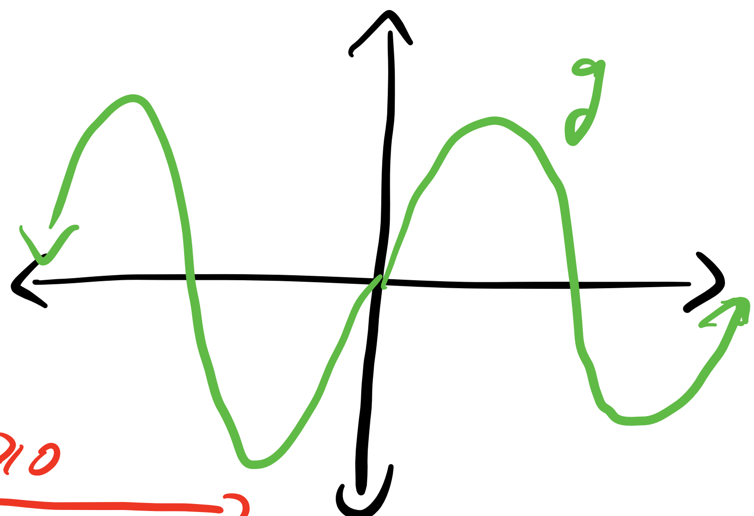
6.) Piecewise functions

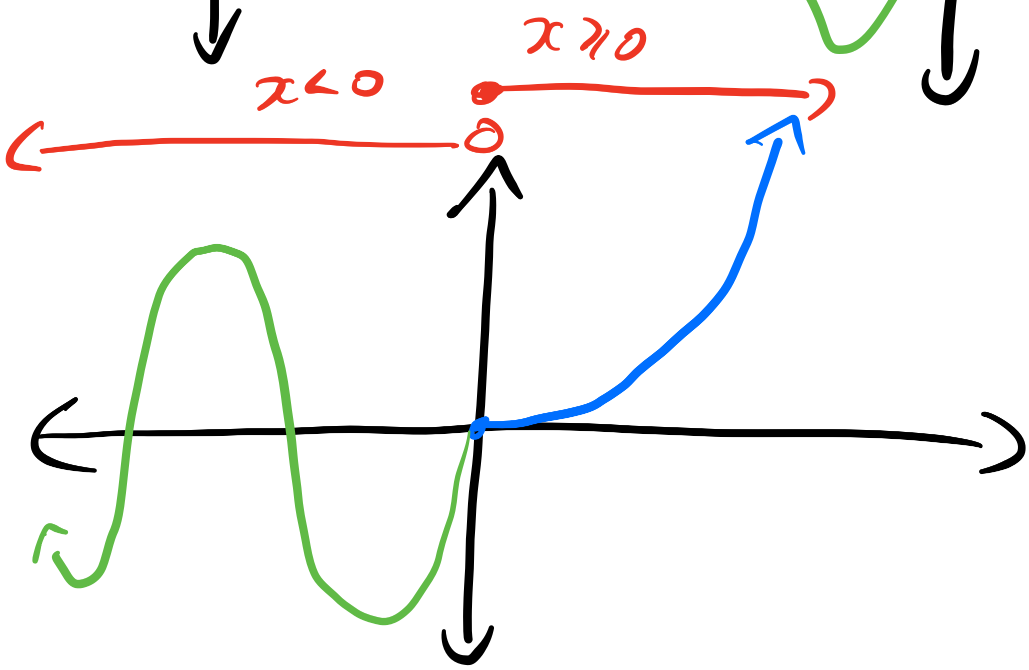
Example: $f(x) = x^2$, $g(x) = \sin(x)$

lets create a new function

$$h(x) = \begin{cases} f(x), & x \geq 0 \\ g(x), & x < 0 \end{cases}$$







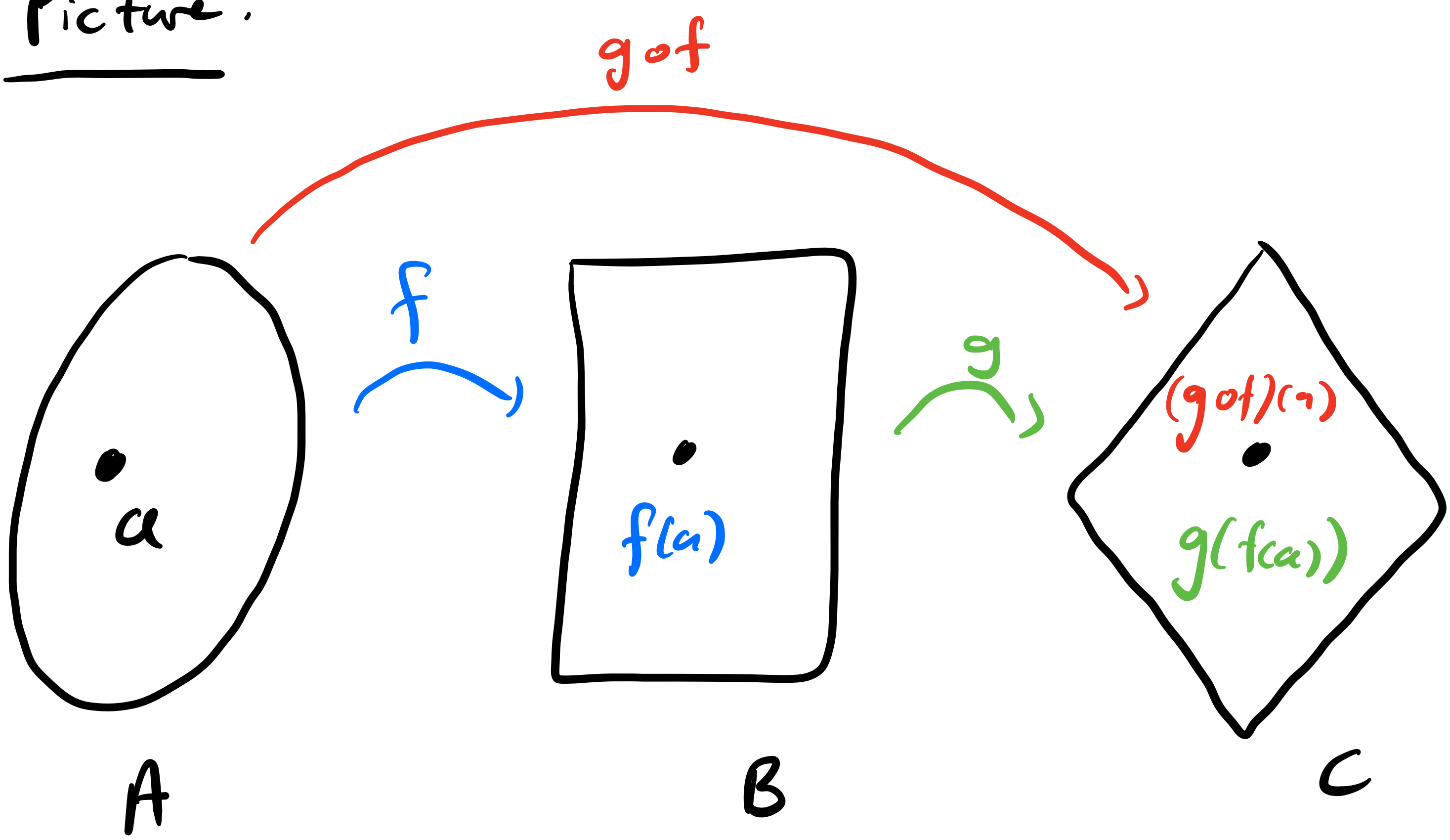
C. Function Composition

def: Given two function $f: A \to B$ onto and $g: B \to C$, we define

$$g \circ f : A \to C \text{ by}$$

$$(g \circ f)(a) = g(f(a))$$

Picture:



Example: (Class example 1) #7

Given $f(x) = \sqrt{x+1}$, $g(x) = \dfrac{x-2}{x-1}$.

Find the domain and range of $f \circ g$.

Solution: $f \circ g(x) = f(g(x))$

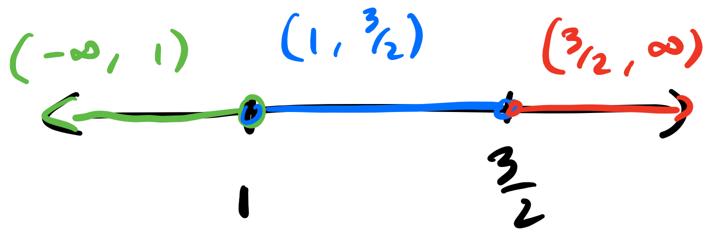
$$= \sqrt{g(x) + 1}$$

$$= \sqrt{\frac{x-2}{x-1} + 1}$$

To find domain we need to determine when

$$\frac{x-2}{x-1} + 1 \geq 0.$$

Well $\dfrac{x-2}{x-1} + 1 = \dfrac{x-2+x-1}{x-1} = \dfrac{2x-3}{x-1}$



- $2x-3=0 \Rightarrow x = \frac{3}{2}$
- $x-1=0 \Rightarrow x=1$

only points where the change sign.

	$(-\infty,1)$	$(1, \frac{3}{2})$	$(\frac{3}{2}, \infty)$
$2x-3$	−	−	+
$x-1$	−	+	+
$\frac{2x-3}{x-1}$	+	−	+

Example : Continued

And we found that $\frac{2x-3}{x-1} \geq 0$ $(+)$
for all the x values in

$$\boxed{(-\infty, 1) \cup [\tfrac{3}{2}, \infty).}$$

We don't include 1 becase the function is not defined at f.

d.) Inverse Functions :

Example : Is there a relationship between the two functions :

- $f(x) = 3x - 2$ and
- $g(x) = \frac{x}{3} + \frac{2}{3}$

well $f(-2) = 3(-2) - 2 = -8$

and $g(-8) = \frac{-8}{3} + \frac{2}{3} = \frac{-6}{3} = -2.$

So $g(f(-2)) = g(-8) = -2.$

And $g(10) = \frac{10}{3} + \frac{2}{3} = 4,$

and $f(4) = 3(4) - 2 = 10$

So $f(g(10)) = f(4) = 10.$

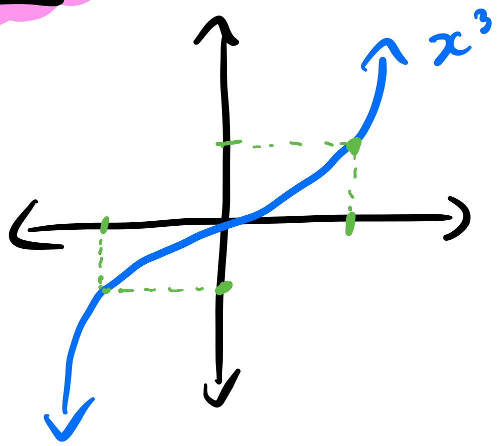
In both cases :

- $g \circ f(x) = x$
- $f \circ g(x) = x$

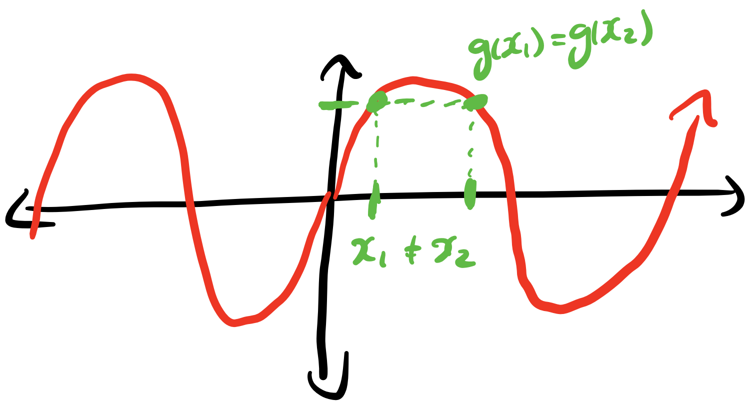
More Inverse functions:

def: We say a function f is <u>one-to-one</u> to mean if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Example: • $f(x) = x^3$ is one-to-one



• $g(x) = \sin(x)$ is <u>not</u> one-to-one on $\mathbb{R}$



If a function f is one-to-one, then it's <u>inverse</u> is the unique function g such that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

Example: Take $f(x) = x^2$ and only consider the domain $[0, +\infty)$

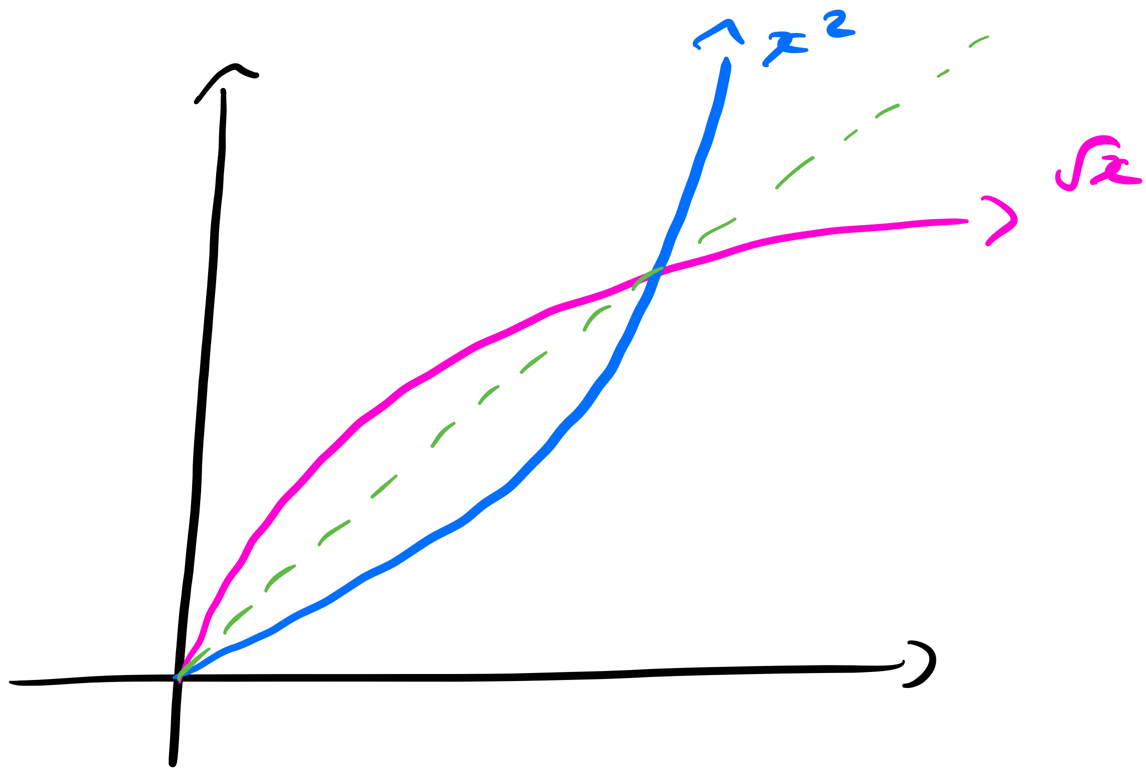
Note $f^{-1}(x) = \sqrt{x}$ sine for $x \geqslant 0$ we have

- $\sqrt{x^2} = x$
- $(\sqrt{x})^2 = x$

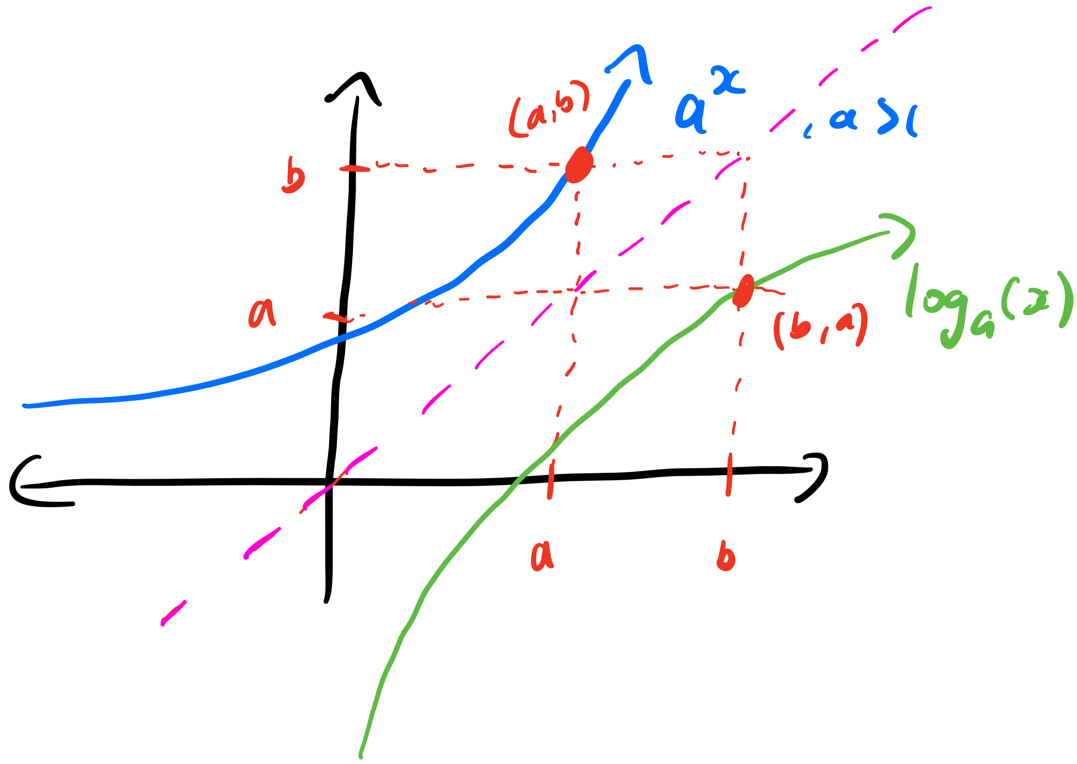
Also note that $f(x) = y \iff f^{-1}(y) = x$.

- $f(3) = 9 \iff f^{-1}(9) = 3$
- $f(4) = 16 \iff f^{-1}(16) = 4$
- $f(5) = 25 \iff f^{-1}(25) = 25$

What should f^{-1} be? Answer: $f^{-1} = \sqrt{x}$.



Example: $f(x) = a^x$ and $g(x) = \log_a(x)$ are inverses of each other. (where $a > 0$, $a \neq 1$)



In particular e^x and $\ln(x)$ are inverses of each other

$$\boxed{\begin{array}{l} \bullet\ e^{\ln(x)} = x \\ \bullet\ \ln(e^x) = x \end{array}}$$

Very important!

$f(x) = e^x$ $\qquad$ $f(x) = y$ iff

$f^{-1}(x) = \ln(x)$ $\qquad$ $f^{-1}(y) = x$

$e^x = y$ iff $\ln(y) = x$

$\ln(x) = y$ iff $e^y = x$

$\ln(z) = y$

$\Leftrightarrow e^y = z$

Example (Finding Inverses)

#2

Find the inverse of $h(x) = \frac{x+4}{2x-5}$

That is find h^{-1} s.t. $h(h^{-1}(x)) = x$

Say $h^{-1}(x) = y$.
Set $h(y) = x$ and solve y.

Then $\frac{y+4}{2y-5} = x \Rightarrow y+4 = 2xy - 5x$

$$\Rightarrow y - 2xy = -5x - 4$$

$$\Rightarrow y(1-2x) = -5x - 4$$

$$\Rightarrow y = \frac{-5x-4}{1-2x}$$

The inverse of $h(x) = \frac{x+4}{2y-5}$ is

$$\boxed{h^{-1}(x) = \frac{-5x-4}{1-2x}}$$

More Practise:

Example: Find the intervals such that

$$\frac{e^x (x-2)}{x^3} \geq 0$$

Example: Solve for x using the properties of logarithms:

$$\ln(3-2x) = 2\ln(1-x)$$

Properties of logarithms

1.) $\frac{\ln(a)}{\ln(b)} = \ln(a-b)$ if $a-b>0$

2.) $\ln(ab) = \ln(a+b)$ if $a+b>0$

3.) $\ln(a^x) = x\ln(a)$

4.) $e^{\ln(y)} = y$, $y>0$

Example (Absolute Values)

a.) Solve the inequality $|5x-2| < 6$.

Solution:

We need $5x-2 < 6$ and $-(5x-2) < 6$

I.e $5x-2 < 6$ and $5x-2 > -6$

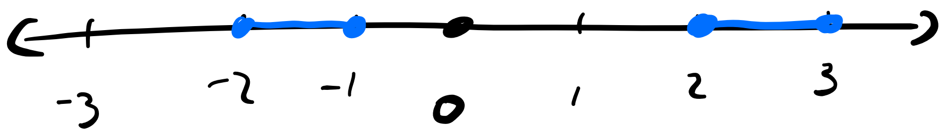
So $-6 < 5x-2 < 6$

$\Rightarrow$ $-4 < 5x < 8$

$\Rightarrow$ $\frac{-4}{5} < x < \frac{8}{5}$

So if $x \in \left(-\frac{4}{5}, \frac{8}{5}\right)$, then $|5x-2| < 6$

(b) Find the interval(s) s.t $2 \le |x| \le 3$.

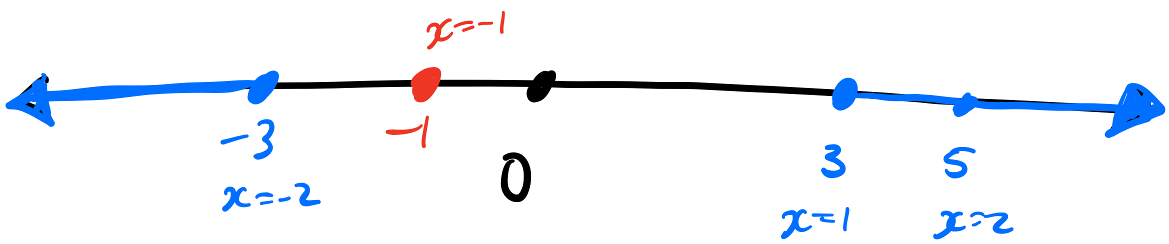


we need $2 \le x \le 3$ or $-3 \le x \le -2$

So $[-3,-2] \cup [2,3]$ is the intervals s.t

$2 \le |x| \le 3$.

c) Find the intervals s.t $|2x+1| \geq 3$



- $2x+1 \geq 3 \Rightarrow x \geq 1$

or

$-(2x+1) \geq 3$

$\Rightarrow 2x+1 \le -3 \Rightarrow 2x \le -4 \Rightarrow x \le \frac{-4}{2} = -2$

So $x \in (-\infty, -2] \cup [1, \infty) \Rightarrow |2x+1| \geq 3$.