Discussion Notes 3

Jan 23, 2024

Last time: (L1–L4)

- Equations and inequalities
- Functions (Domain, range, composition, inverses)
  - Types of functions.
  - Building new functions from old.

Today: (L5–L6)

- Evaluating limits.
  - Infinite limits, limits to infinity.
  - One sided limits
  - Existence of limits
  - Algebraic Manipulations
  - Absolute values

- Algebraic limit laws.
- Squeeze Theorem.
- Vertical and horizontal asymptotes

Example 1 (Infinite limits)

Evaluate $\displaystyle\lim_{x \to \infty} \frac{27x^2 - 6x + 3}{2x + 4x^2 + 17} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$

$$= \lim_{x \to \infty} \frac{27 - \frac{6}{x} + \frac{3}{x^2}}{\frac{2}{x} + 4 + \frac{17}{x^2}}$$

$$= \frac{\displaystyle\lim_{x \to \infty} 27 - \cancel{\frac{6}{x}}^{\to 0} + \cancel{\frac{3}{x^2}}^{\to 0}}{\displaystyle\lim_{x \to \infty} \cancel{\frac{2}{x}}^{\to 0} + 4 + \cancel{\frac{7}{x^2}}^{\to 0}}$$, Quotient limit law

$$= \frac{27}{4}$$

Example 2

$x^{-1} = \frac{1}{x}$

Evaluate $\lim_{x \to \frac{1}{3}} \frac{x^{-1} - 3}{x - \frac{1}{3}}$

$$= \lim_{x \to \frac{1}{3}} \left(\frac{1}{x} - 3\right) \cdot \frac{1}{x - \frac{1}{3}}$$

$$= \lim_{x \to \frac{1}{3}} \frac{1 - 3x}{x} \cdot \frac{1}{\frac{3x - 1}{3}}$$

$$= \lim_{x \to \frac{1}{3}} \frac{1 - 3x}{x} \cdot \frac{3}{3x - 1}$$

$$= \lim_{x \to \frac{1}{3}} \frac{-3}{x}$$

$$= -3/\tfrac{1}{3} = \boxed{-9}$$

Example 3: (multiply by conjugate)

Evaluate

$$\lim_{x \to 1} \frac{\sqrt{10x-9} - 1}{x-1}$$

conjugate

We get
$$\lim_{x \to 1} \frac{\sqrt{10x-9} - 1}{x-1} \left(\frac{\sqrt{10x-9} + 1}{\sqrt{10x-9} + 1} \right)$$

$$= \lim_{x \to 1} \frac{(10x-9) - 1}{(x-1)(\sqrt{10x-9} + 1)}$$

$$= \lim_{x \to 1} \frac{10x - 10}{(x-1)(\sqrt{10x-9} + 1)}$$

factor $10x - 10 = 10(x-1)$ and cancel $(x-1)$'s

$$= \lim_{x \to 1} \frac{10}{\sqrt{10x-9} + 1}$$

evaluate

$$= \frac{10}{\sqrt{10-9} + 1}$$

$$= \boxed{5}$$

Example 4:

Evaluate $\lim_{x \to \infty} \frac{\sqrt{9x^4+1}}{x^2-3x+5}$

Factor out x^2

To get $\lim_{x \to \infty} \frac{\sqrt{(9+\frac{1}{x^4})(x^4)}}{(1-\frac{3}{x}+\frac{5}{x^2})\,x^2}$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{9+\frac{1}{x^4}}\right)\cancel{x^2}}{\left(1-\frac{3}{x}+\frac{5}{x^2}\right)\cancel{x^2}}$$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

$$= \frac{\sqrt{\lim_{x \to \infty} 9 + \cancel{\frac{1}{x^4}}^{\to 0}}}{\lim_{x \to \infty} 1 - \cancel{\frac{3}{x}}^{\to 0} + \cancel{\frac{5}{x^2}}^{\to 0}}$$

, limit laws

$$= \frac{\sqrt{9}}{1}$$

$$= \boxed{3}$$

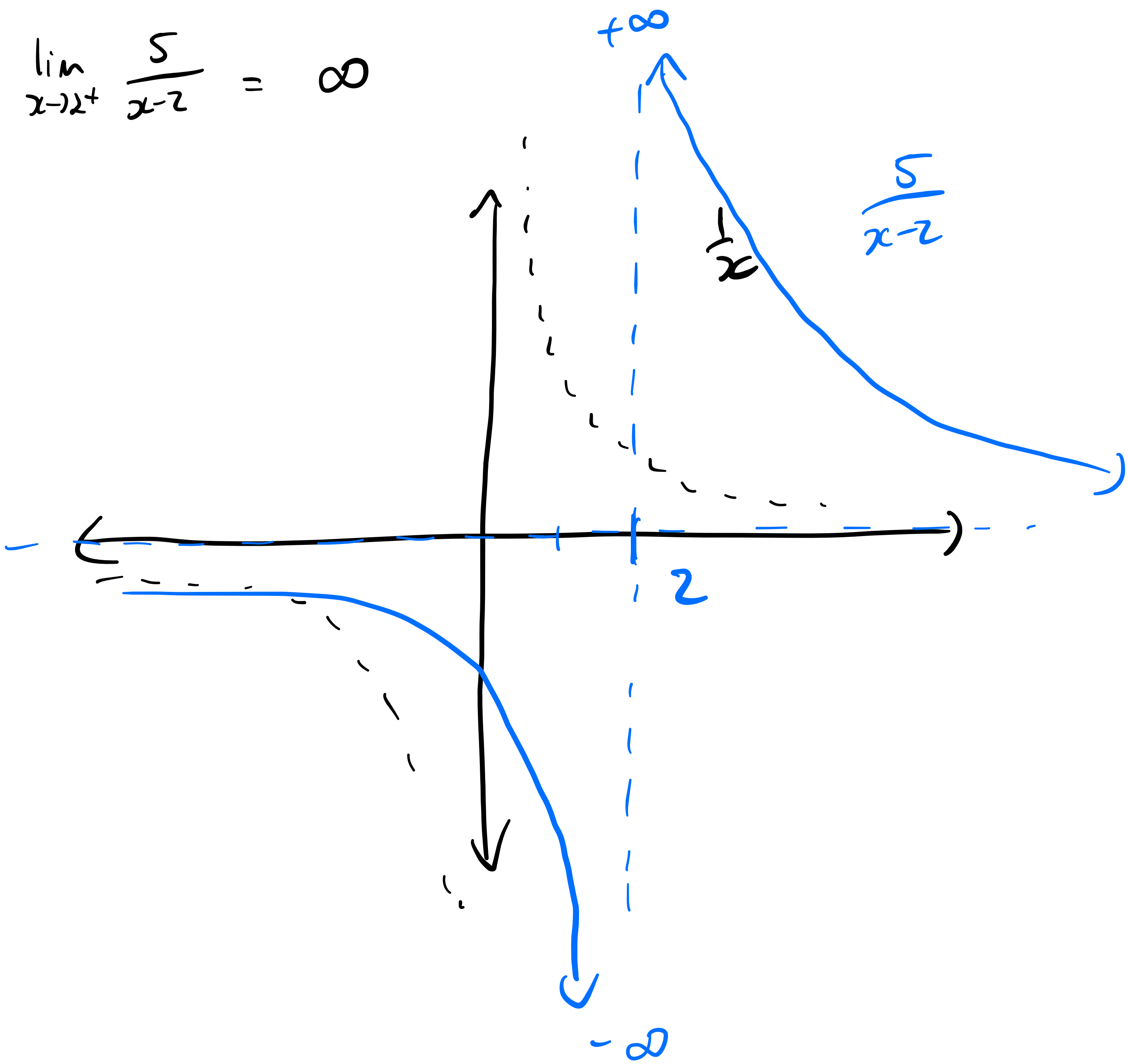
Example 5: Evaluate the following limit.

$$\lim_{x \to 2} \frac{5}{x-2}$$

Well let's look at one sided limits:

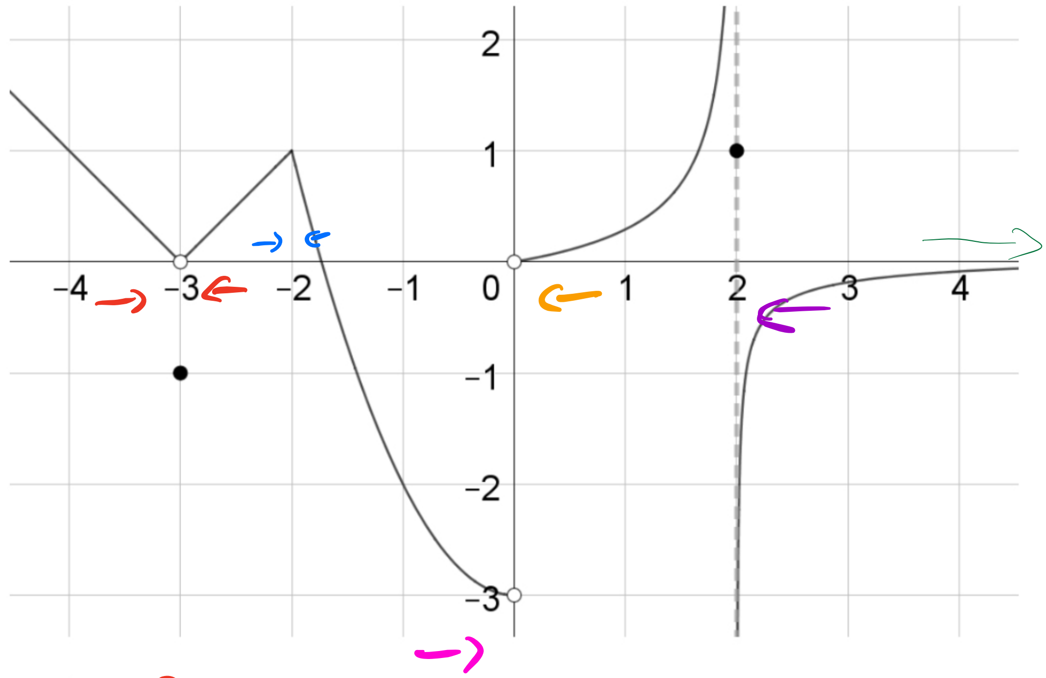
- $\lim_{x \to 2^-} \frac{5}{x-2} = -\infty$

- $\lim_{x \to 2^+} \frac{5}{x-2} = \infty$



Example : Evaluating limits with Graphs

5. Use the graph of the function $f(x)$ below to evaluate the following limits; use the symbols ∞, $-\infty$, or DNE when appropriate:



(a) $\lim_{x\to -3} f(x) =$ 0

(b) $\lim_{x\to -2} f(x) =$ 1

(c) $\lim_{x\to 0^-} f(x) =$ -3

(d) $\lim_{x\to 0^+} f(x) =$ 0

(e) $\lim_{x\to 0} f(x) =$ DNE since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

(f) $\lim_{x\to 2^+} f(x) =$ $-\infty$

(g) $\lim_{x\to \infty} f(x) =$ 0

Using limits to find Horizontal & Vertical asymptotes

Example: Find all the vertical and horizontal asymtotes of :

$$k(x) = \frac{-3x^2+6x}{x^2-2x+1}$$

what is a horizontal ?

$$= \frac{-3(x-2)}{(x-1)(x-1)}$$

$\lim_{x \to 1}$

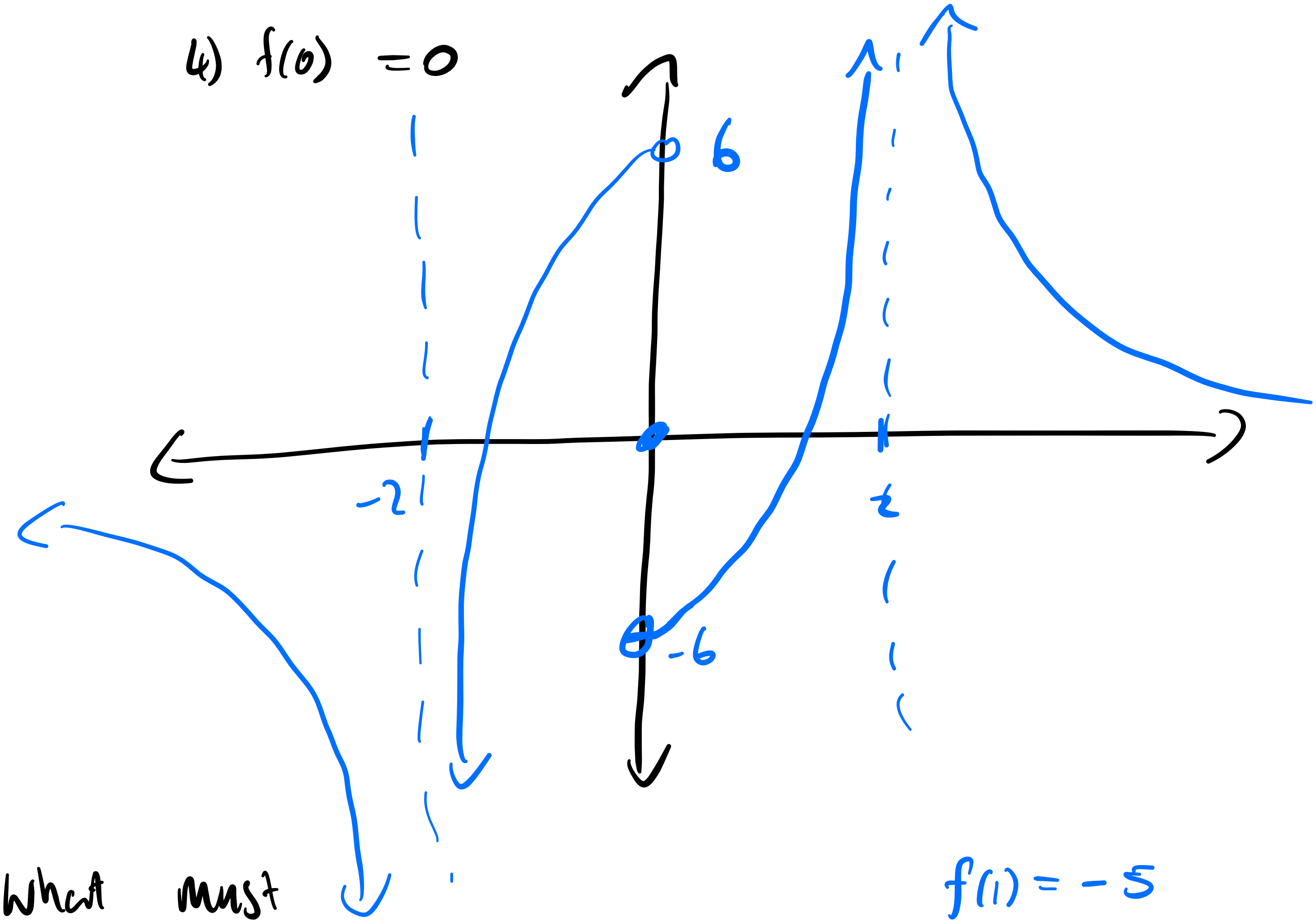
$\lim_{x \to +\infty}$

$\lim_{x \to -\infty}$

Notes from lecture:

Example: Sketch a function f s.t.

1) f is odd $\quad (-f(x) = f(-x), \forall x)$
2) $\lim_{x \to 0^+} f(x) = -6$
3) $\lim_{x \to 2} f(x) = \infty$
4) $f(0) = 0$



What must

- $\lim_{x \to 0^-} f(x)$? $= 6$
- $\lim_{x \to -2} f(x)$? $= -\infty$

$f(1) = -5$

$f(-1) = -(-5) = 5$

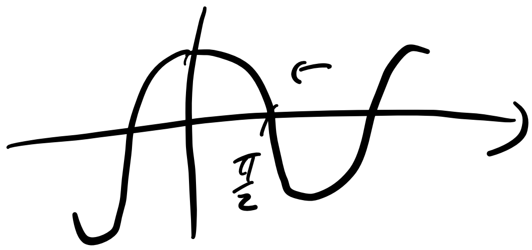
Example: $\lim_{x \to \frac{\pi}{2}^+} \sec(x)$.

$\frac{1}{\sin}$	csc
$\frac{1}{\cos}$	sec
$\frac{1}{\tan}$	cot

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{1}{\cos(x)}$$

$$= \frac{1}{\text{“}0^-\text{”}}$$

$$= -\infty$$



- $\lim_{x \to -\frac{\pi}{2}^-} \sec(x) = -\infty$.