Discussion Notes 6

Feb 13, 2024

Last Time (L11)

- Power rule, exponential rule
- Sum rule, constant multiples

Today (L12 ~ L14)

- Product Rule and Quotient Rule
- Rates of Change (Higher order derivatives)
- Trigonometric derivatives.

Differentiation Rules (so far)

- let $f, g : I \to \mathbb{R}$ differentiable, c a constant.

① Constant Multiple Rule

- $\frac{d}{dx} cf(x) = c f'(x)$

② Sum Rule

- $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

③ Power Rule

- $\frac{d}{dx} x^n = nx^{n-1}$

④ Product Rule

- $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

⑤ Quotient Rule

- $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

⑥ Two trigonometric derivatives (with these two you can find the rest)

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$

⑦ Exponents : $(b > 0, b \neq 1)$

- $\frac{d}{dx} b^x = b^x \ln(x)$

⑧ Other

- $\frac{d}{dx} \ln(x) = \frac{1}{x}$

Higher Order Derivatives

Given a function a differentiable function f.
We can find is derivative f' which is another function. If f' is differentiable, then we can find it's derivative called the <u>second order derivative</u> of f.

Denoted f'' or $\frac{d^2f}{dx^2}$

Example: $f(x) = \sin(x) + x^5 + 3x$

First derivative : $f'(x) = \cos(x) + 5x^4 + 3$

Second derivative : $f''(x) = -\sin(x) + 20x^3$

$f'''(x) = -\cos(x) + 60x^2$

$\vdots$

Applications

- **Physics** :- Position, velocity, acceleration
  - Dicplacement, total distance
  - Speeding up, slowing down

- **Biology** : Agricultural output or Population
  - Increase / decrease

- **Economics** : • Cost, Marginal cost
  - Revenue = price • # units sold
  - Profit = Revenue − cost, Marginal profit

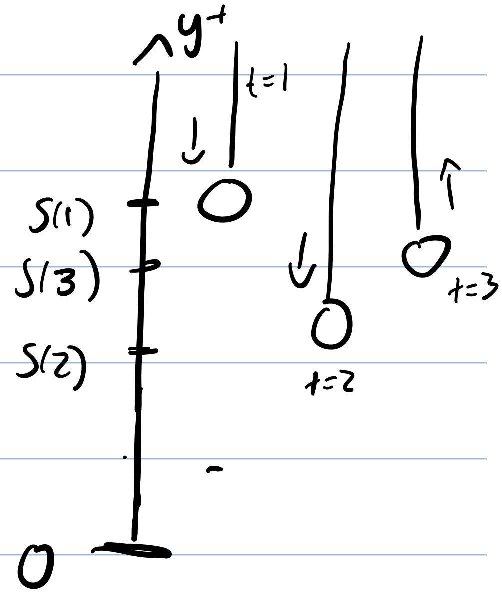
  * When you here Marginal think derivatives.

- Slopes of tangent lines

Example: (Physics Examples)

The vertical position of a yo-yo is given (meters) by

$$\boxed{S(t) = \sqrt{3}\sin(t) - \cos(t)}$$



a) What is the displacement from time $t=0$ to $t=\frac{\pi}{2}$

- Displacement $= S(t_f) - S(t_i)$

$$= S(\tfrac{\pi}{2}) - S(0)$$

$$= \sqrt{3} - 1 \text{ m}$$

b) What is the total distance travelled, from $t=0$ to $t=\pi$?

- At what points did we turn around?

  Every time $v(t)=0$ where $v(t) = s'(t) = \sqrt{3}\cos(t) + \sin(t)$

- So to find the points where we turn around we need to solve

$$\sqrt{3}\cos(t) + \sin(t) = 0$$

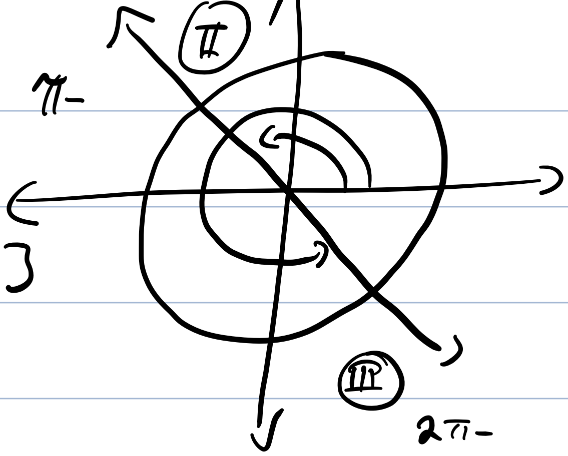
$$\Rightarrow \quad \sin(t) = -\sqrt{3}\cos(t)$$

$$\Rightarrow \quad \tan(t) = -\sqrt{3}$$

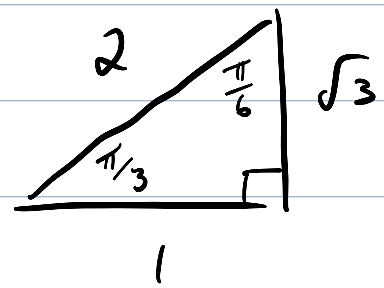
- $\tan(\frac{\pi}{3}) = \sqrt{3}$

$\Rightarrow t = \pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$

or $t = 2\pi - \frac{\pi}{3} = \boxed{\frac{5\pi}{3}} \notin [0, \pi]$



So we change direction possibly at $\frac{2\pi}{3}$



Then Total distance:

$$= \underbrace{\left| S\left(\tfrac{2\pi}{3}\right) - S(0) \right|}_{A} + \underbrace{\left| S(\pi) - S\left(\tfrac{2\pi}{3}\right) \right|}_{B} = 3 + 1 = 4 \text{ m.}$$

$$A = \left| S\left(\tfrac{2\pi}{3}\right) - S(0) \right| = |2 - (-1)| = 3$$

$$B = \left| S(\pi) - S\left(\tfrac{2\pi}{3}\right) \right| = |1 - 2| = 1$$

c) On which intervals is the yo-yo speeding up and slowing down? From $t = 0$ to $t = \pi$.

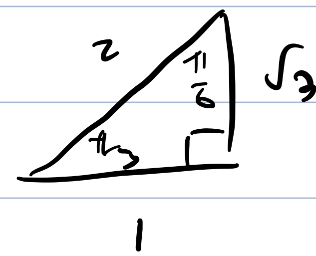
- Speeding up : $v > 0, a > 0$ or $v < 0, a < 0$.
- Slowing down : $v > 0, a < 0$ or $v < 0, a > 0$

We have - velocity : $v(t) = \sqrt{3}\cos(t) + \sin(t)$

- acceleration : $a(t) = -\sqrt{3}\sin(t) + \cos(t)$

• $v(t) = 0 \Rightarrow t = \frac{2\pi}{3}$

v	$+$	$-$
t	$[0, \frac{2\pi}{3})$	$(\frac{2\pi}{3}, \pi]$



• $a(t) = 0 \Rightarrow -\sqrt{3}\sin(t) + \cos(t) = 0$

$\Rightarrow \cos(t) = \sqrt{3}\sin(t)$

$\Rightarrow \tan(t) = \frac{1}{\sqrt{3}}$

$\Rightarrow t = \frac{\pi}{6} \pm k \cdot \pi$

Since on $[0, \pi]$ we have $t = \frac{\pi}{6}$

a	$+$	$-$
t	$[0, \frac{\pi}{6})$	$(\frac{\pi}{6}, \pi]$

Speeding up : $[0, \frac{\pi}{6}) \cup (\frac{4\pi}{6}, \pi]$

Slowing down : $(\frac{\pi}{6}, \frac{4\pi}{6})$

• constant v -

Trigonometric Derivaties

$\frac{d}{dx}\sin x = \cos(x)$

$\frac{d}{dx}\csc x = -\csc(x)\cot(x)$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\sec x = \sec(x)\tan(x)$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\cot x = -\operatorname{cosec}^2(x)$

Inverse Trigonometric derivates

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Example: Given $g(x) = x^{\frac{1}{5}}(x-1)^{3/5}$ find the domain of g'.

Step 1: Use the product rule to find g'.

$$g' = \frac{1}{5}x^{-\frac{4}{5}} \cdot (x-1)^{3/5} + x^{\frac{1}{5}} \cdot \frac{3}{5}(x-1)^{-2/5}$$

$\neq 0$

$\Rightarrow x \neq 0$

$\neq 0 \Rightarrow x \neq 1$

Step 2: Determine where g' is defined.

Since $x^{-\frac{4}{5}} = \frac{1}{x^{4/5}}$ and $(x-1)^{-2/5} = \frac{1}{(x-1)^{2/5}}$

we know the domain of g' is

$(-\infty, 0) \cup (0,1) \cup (1, \infty)$.

Example: (Q14, Fall exam 2)

Given $f(\theta) = \cos(\theta) + \sin^2(\theta)$, how may horizontal tangent lines does f hae?