Discussion Note 7

Feb 20, 2023

Last Time

- Product Rule, Quotient Rule
- Higher Order Derivatives
- Trigonometric Derivatives

Today

- Chain Rule.
- Implicit Differentiation.

Why do we need the Chain Rule?

- The more functions we can differentiate the better
- Using composition we can build more functions.

Why do we need implicit differentiation?

- We want the rate of change of curves that cannot be represented by functions.

Chain Rule: A method for finding the derivatives of the composition of two functions.

Example: How do we find derivatives of the following.

- $\ln(\sin(x))$?
- $\sin(\cos(x))$? (This is not the product $\sin(x) \cdot \cos(x)$)

Sum, product, quotient, exponential rules won't help. We need a new tool!

Chain Rule!: If f differentiable at $g(x)$, and g diff at x

$$\boxed{\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big) \cdot g'(x)}$$

Do:

a, c, cos, impl, d

Example: [Chain Rule - Find derivatives]

a) $g(z) = \cos(\sin(z) + z^2)$

b) $h(u) = \sec(u^2 - u)$ $\qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

c) $f(t) = \sin(t^3 \cdot e^{-6t})$

d) Determine where $H(\omega) = (\omega^2 - 1)e^{2-\omega^2}$ is increasing and decreasing.

e) $g(v) = \sqrt{\dfrac{e^v}{7+2v}}$

Solutions:

a) $$\frac{dg}{dz} \overset{\text{Chain Rule}}{=} -\sin(\sin(z) + z^2)\frac{d}{dz}(\sin(z) + z^2)$$

sum rule

$$= -\sin(\sin(z) + z^2) \cdot (\cos(z) + 2z)$$

b) $$\frac{dh}{dz} = \sec(u^2 - u)\tan(u^2 - u) \cdot \frac{d}{du}(u^2 - u)$$

$$= \sec(u^2 - u)\tan(u^2 - u)(2u - 1)$$

c) $\frac{df}{dt} = \cos(t^3 \cdot e^{-6t}) \cdot \frac{d}{dt}(t^3 \cdot e^{-6t})$

↳ chain rule

↓ product rule

$$= \cos(t^3 \cdot e^{-6t}) \cdot \left(3t^2 \cdot e^{-6t} + t^3 \cdot e^{-6t} \cdot (-6)\right)$$

d) To determine where H is increasing and decreasing we need $H'(w) \geq 0$ and $H'(w) \leq 0$.

$$H'(t) = \frac{dH}{dw} = \frac{d}{dw}\left((w^2-1)(e^{2-w^2})\right)$$

↓ product Rule

$$= (2w)\cdot(e^{2-w^2}) + (w^2-1)\cdot\left(e^{2-w^2} \cdot \frac{d}{dw}(2-w^2)\right)$$

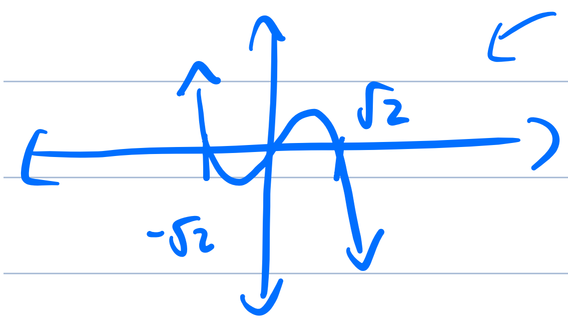
$$= (2w)(e^{2-w^2}) + (w^2-1)\left(e^{2-w^2} \cdot (-2w)\right)$$

$$= \underbrace{e^{2-w^2}}_{\text{Always } >0} \underbrace{\left(2w - 2w(w^2-1)\right)}_{\text{This will control the sign}}$$

Hence look at $2w - 2w^3 + 2w = -2w^3 + 4w$

$$= (-2w)(w^2-2)$$

$$= (-2w)(w-\sqrt{2})(w+\sqrt{2})$$



Implicit differentiation :

Example 1 : $x^2y - 4x = y^3$. Find y' (or $\frac{dy}{dx}$)

Notice that y is a function of x. To make things more clear denote $y = f(x)$. Then we have

$$x^2 \cdot f(x) - 4x = f(x)^3$$

$$\Rightarrow \frac{d}{dx}\left(x^2 \cdot f(x) - 4x \right) = \frac{d}{dx} f(x)^3$$

$$\Rightarrow \quad 2x \cdot f(x) + x^2 \cdot f'(x) - 4 = 3 f(x)^2 \cdot f'(x)$$

And since $y' = f'(x)$ (same thing just different notation)

$$2x \cdot y + x^2 y' - 4 = 3y^2 \cdot y'$$

$$x^2 y' - 3y^2 y' = 4 - 2xy$$

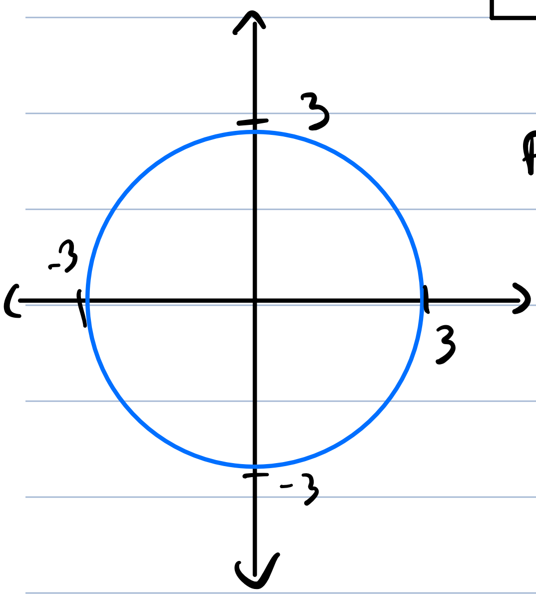
$$y'(x^2 - 3y^2) = 4 - 2xy$$

$$\boxed{y' = \frac{4 - 2xy}{x^2 - 3y^2}}$$

$0 =$ -

Example 2: Given an equation of a circle centered at the origin with radius 3.

$$\boxed{x^2 + y^2 = 9}$$



Find the slope of the tangent line

a) Explicitly.

b) Implicitly.

c) Find $\frac{d^2y}{dx^2}$

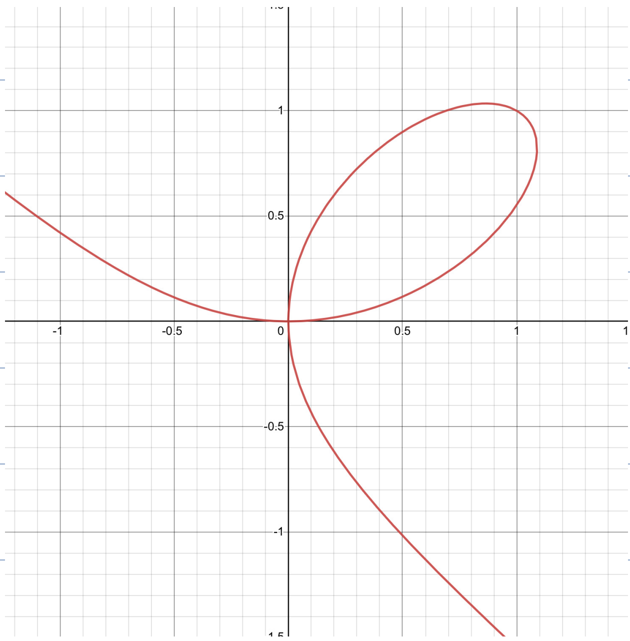
d) Find $\underline{\underline{y'(3)}}$

explain what that means.

Examples: (Implicit Differentiation)

Given $6x^3 + 7y^3 = 13xy$

Find $\frac{dy}{dx}$ or y':



Plot of all the points (x,y) satisfying the equation $6x^3+7y^3 = 13xy$

Sol: $18x^2 + 21y^2 \cdot \frac{dy}{dx} = 13y + 13x \cdot \frac{dy}{dx}$

$\Rightarrow \quad 21y^2 \cdot \frac{dy}{dx} - 13x\frac{dy}{dx} = 13y - 18x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{13y - 18x^2}{21y^2 - 13x}$

Example 3 Suppose both x and y are functions of a variable t. Then consider

$$x^3y^6 + e^{1-x} - \cos(5y) = y^2$$

Differentiate the following equation with respect to t.

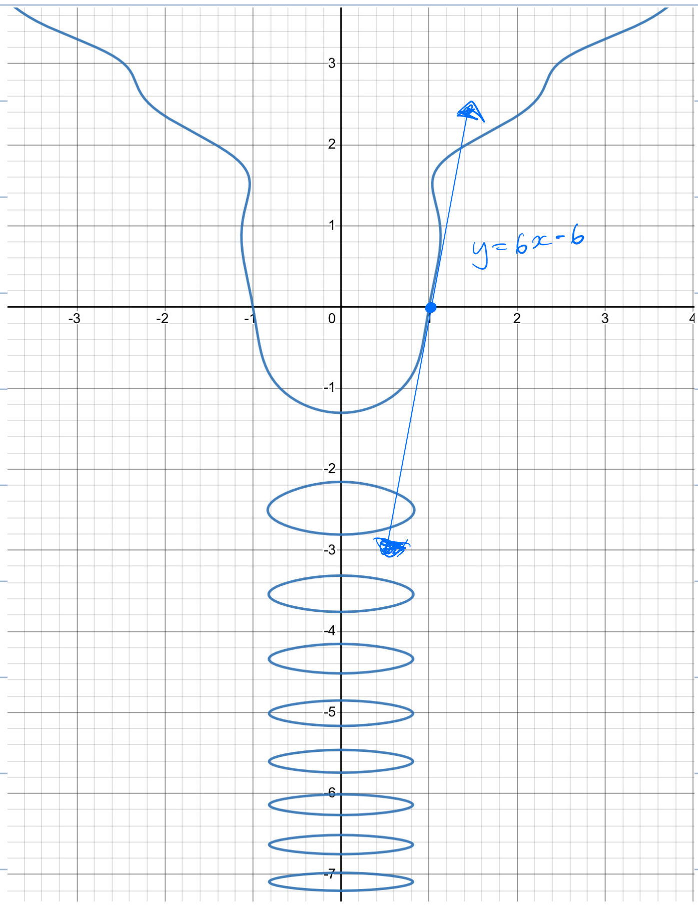
Sol: $$\frac{d}{dt}\left(x^3y^6 + e^{1-x} - \cos(5y)\right) = \frac{d}{dt}y^2$$

$$\Rightarrow 3x^2 \cdot \frac{dx}{dt} \cdot y^6 + x^3 \cdot 6y^5 \frac{dy}{dt} + e^{1-x} \cdot \frac{d}{dt}(1-x) + \sin(5y)\, 5\frac{dy}{dt} = 2y \cdot \frac{dy}{dt}.$$

$$\Rightarrow 3x^2y^6 \cdot \frac{dx}{dt} + 6x^3y^5 \cdot \frac{dy}{dt} - e^{1-x} \cdot \frac{dx}{dt} + 5\sin(y)\frac{dy}{dt} = 2y \cdot \frac{dy}{dt}$$

Example 4: Given $2\cos(y^2) = 3x^2 - e^y$

- a) Use implicit differentiation to find y'.
- b) Find the equation of the tangent line to the curve at the point $(1,0)$.



Plotting all the points (x,y) that satisfy the given equation.

Sol: $\frac{d}{dx}\, 2\cos(y^2) = \frac{d}{dx}\left(3x^2 - e^y\right)$

$$\Rightarrow -2\sin(y^2)\cdot 2y \cdot \frac{dy}{dx} = 6x - e^y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(e^y - 2\sin(y^2)\cdot 2y\right) = 6x$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x}{e^y - 2\sin(y^2)\cdot 2y}$$

Find equation of the tangent line at the point (1, 0).

Find $\left.\frac{dy}{dx}\right|_{x=0} = \frac{6(1)}{e^0 - 2\sin(0^2)\cdot 2(0)} = 6$

$\quad ''$

$y'(0)$

Then $y - y_1 = y'(0)(x - x_1)$

$y - 0 = 6(x - 1)$

$$\boxed{y = 6x - 6}$$