Discussion Notes 9

March 19, 2024

Last time:

- Linear Approximation
- Differentials

Today:

- Extreme Values and Extreme Value Theorem.
- Mean Value Theorem, Rolle's Thm
- First Derivative Test

Important definitions

- Critical point.
- Abs. Maximum, Abs. min.
- Local Max and local min.

Extreme Values.

- Increasing and decreasing.

• **Critical Point:** we say a number c is a critical point of a function f to mean

1) $f'(c) = 0$ or 2) $f'(c)$ does not exists.

• **Absolute Maximum and Absolute Minimum.**

We say a function has an absolute maximum at the number x_0 on the interval I to mean

$$f(x_0) \geq f(x) \quad \text{for all}$$

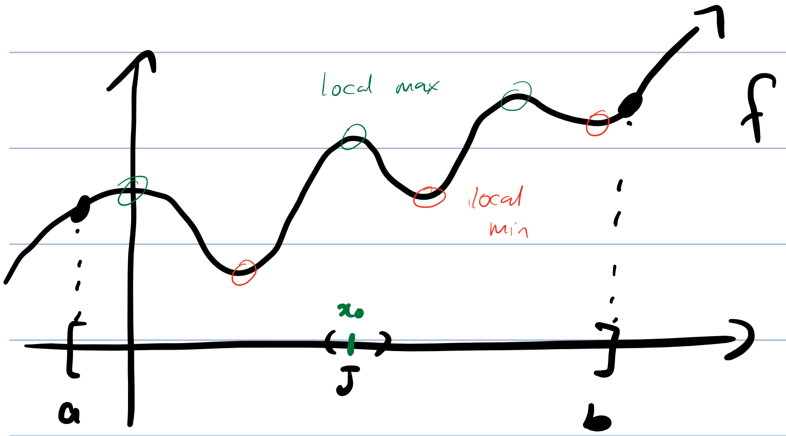
numbers x in the interval I.

Same idea for absolute minimum

local maximum and local minimum

Given a function f and an interval I. We say f has a local maximum at a number x_0 in I to mean we can find a smaller interval J contained in I s.t x_0 has an absolute maximum at x_0 on J.

- Local max and min :



$I = [a, b]$

- On a necessary condition for local extrema

Thm (Fermat)

If f has a local extrema at c and $f'(c)$ exists, then $f'(c) = 0$.

- converse :
  - If $f'(c)$ exists and $f'(c) \neq 0$, then f does not have a local extrema at c.

Thm: (Extreme Value Thm)

If f is a function continuous on $[a,b]$ then f achieves extreme values on $[a,b]$.

Thm (Rolle's Thm)

If $f: [a,b] \to \mathbb{R}$ continuous on $[a,b]$, and differentiable on (a,b), and $f(a) = f(b)$, then there exists $c \in (a,b)$ s.t $f'(c) = 0$

Thm (Mean Value thm)

If f a function s.t

- continuous on $[a,b] \subseteq \mathbb{R}$
- differentiable in (a,b)

Then there exists a c in (a,b) s.t

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Mean Value Thm

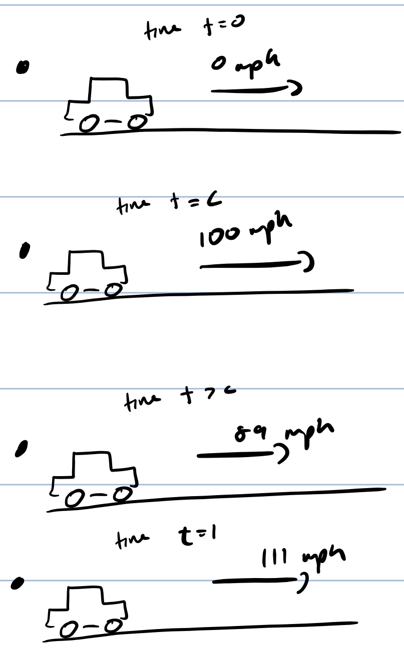
Thm: If $f:[a,b] \to \mathbb{R}$ continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Intuitively: If f represents velocity, it means over a time interval $[a,b]$, the average velocity and instantaneous velocity coincide at at least one point in time

- Think about it like this:

- If average 100 mph from time 0 to time 1 hour, then there must have been at least one c in time $(0,1)$ such that we drove exactly 100 mph at time c.



Example: Use the mean Value theorem to show that for any real numbers $a<b$ we have $-1 \le \frac{\cos(b)-\cos(a)}{b-a} \le 1$.

- To use the MVT we need 1) a function and 2) an interval.
- Consider the interval $[a,b]$ and function $f(x) = \cos(x)$.

– Why can we apply MVT?

⋮

– Then there exists c in (a,b) st $f'(c) = \frac{f(b)-f(a)}{b-a}$.

That is $-\sin(c) = \frac{\cos(b)-\cos(a)}{b-a}$.

Then $\left|\frac{\cos(b)-\cos(a)}{b-a}\right| = |-\sin(c)| \le 1$

$\Rightarrow \quad -1 \le \frac{\cos(b)-\cos(a)}{b-a} \le 1$, as required.

Exercise 1: $f(x) = x^{\frac{2}{3}} - 2$, continuous on $[-1,1]$.

Note that $f(-1) = \sqrt[3]{(-1)^2} - 2$
$= \sqrt[3]{1^2} - 2$
$= f(1)$.

But $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} \neq 0$ for any $x \in [-1,1] \setminus \{0\}$.

This does not contradict Rolle's thm because f is not differentiable on $(-1,1)$ since $f'(0)$ DNE.

Exercise 2: If $f(2) = -2$, $f'(x) \geq 1$ for x in $[2,5]$

How small can $f(5)$ be?

Suppose f satisfy hypothesis of MVT.

Then there exists c in $(2,5)$ st

$$f'(c) = \frac{f(5)-f(2)}{5-2} = \frac{f(5)-f(2)}{3}$$

Then $f'(c) \geq 1 \Rightarrow \frac{f(5)-f(2)}{3} \geq 1$

$\Rightarrow f(5) \geq 3 + f(2)$

$\Rightarrow f(5) \geq 3 - 2 = 1$.

So $f(5)$ at least 1.

Exercise 4: Does there exists a function f such that

- $f(0) = -1$
- $f(2) = 4$
- $f'(x) \leq 2$ for evey x in $[0,2]$?

How to approach?

- Either find an example.
- OR explain why it cannot happen.

(look for necessy conditions)

let's say such an f exist. Then f is differentiable on $[0,2]$ and so continuous on $[0,2]$. So Mean Value Theorem applies.

Meaning there exist c in $(0,2)$ st

$$f'(c) = \frac{f(2) - f(0)}{2}$$

But then $\frac{f(2) - f(0)}{2} = \frac{4-(-1)}{2} = \frac{5}{2}$

But $f'(c) \leq 2$, which canot happen since $2 < \frac{5}{2}$.

So no so f exists.

Extreme Value Question:

Example 2: Given, $f(x) = \dfrac{x^2}{x^2+3}$ (see quiz 9)

- $g(x) = e^{2x} + e^{-x}$

For f and g, find the following:

a) Intervals on which they are increasing/decreasing
b) Local minima and local maxima
c) Intervals of concavity and inflection points.