Discussion 8

Oct 15, 2024

Last Time

- Sequences
- Series (Geometric series, Telescopic Series)

Today

- Summing Series
- Integral Test
- Direct Comparison

In Calculus I:

- Given a sequence (a_n), determine

$\lim_{n\to\infty} a_n$ exists? Value if exists?

In Calculus II:

- Given a sequence (a_n), determine

$$\underbrace{\lim_{n\to\infty} \sum_{k=0}^{n} a_n}_{S_n \text{ - partial sum}} = \lim_{n\to\infty} S_n$$

Converge? if so to what?

So something like the algebraic law for sequences carries directly over;

Thm:

Recall:

NB!

1) Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x}, & \text{if } x \\ \text{Div}, & x \end{cases}$$

2) Telescopic Series

For (b_n) sequen,

then $\sum_{n=0}^{\infty}(b_n - b_{n+1})$ converg iff (b_n) converg.

3) P-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv}, & p>1 \\ \text{div}, & p \le 1 \end{cases}$$

Thm: (The Integral Test)

Let $f : [1, \infty) \to \mathbb{R}$ function s.t

1) Continuous
2) Non-negative
3) Decreasing (Non-increasing)

Let $a_n := f(n)$.

Then $\sum_{n=1}^{\infty} a_n$ converge iff $\int_1^{\infty} f(x)\,dx$ converges (finite)

Thm (Test for divergence)

Let $(a_n) = (a_1, a_2, \ldots)$ sequence of numbers.

If $\sum_{n=0}^{\infty} a_n$ converge, then $(a_n) \to 0$

Conversely: If $(a_n) \not\to 0$, then $\sum_{n=0}^{\infty} a_n$ diverge

Thm (p-series) – Converge if $p>1$
– Div if $p\le 1$.

Then series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\begin{cases}\text{conv}, & p>1\\ \text{div}, & p\le 1\end{cases}$$

* Adding $\ln(n)^q$, helps us for case $p=1$ and $q>1$ to get convergence.

Thm (Generalized p-series)

$$\sum_{n=2}^{\infty}\frac{1}{n^p\ln(n)^q}$$

– Converges $p>1$, or $p=1$, $q>1$
– Diverges $p<1$, or $p=1$, $q\le 1$

Thm: (Direct Comparison)

Let (a_n), (b_n) sequences of positive numbers.

And suppose $a_n\le b_n$ for all n.

1) If $\sum a_n$ diverge, then $\sum b_n$ div.

2) If $\sum b_n$ converge, then $\sum a_n$ conv.

Corollary:

If (a_n), (b_n) all series of positive terms.

and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$.

Then $\sum a_n$ conv. iff $\sum b_n$ converg.

(Behavior of sequences the same means behavior of series the same).

$$-\varepsilon < \frac{a_n}{b_n} - L < \varepsilon$$

$$L-\varepsilon < \frac{a_n}{b_n} < L+\varepsilon$$

$$b_n(L-\varepsilon) < a_n < b_n(L+\varepsilon)$$

Examples.

Example 1: (Determine conv)

① $\sum \frac{4+\cos(n)}{n}$

• div
direct comp
$\sum \frac{3}{n}$, all $\geq$ 0 [illegible]

② $\sum \frac{4-\cos(n)}{2^n}$

• conv, direct comp,
≥ 0 all [illegible]

③ $\sum \frac{1+4\cos(n)}{n^2}$

. inconclusive, since
terms < 0 and > 0.

Geom
Comp test

Example 2

Does $\sum_{n=1}^{\infty} \frac{n}{n^3+5}$ converge?

Yes, by direct comparison

$\frac{n}{n^3+5} < \frac{n}{n^3} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ conv.

Example 3: $$\sum_{n=1}^{\infty} \frac{5n\sqrt{n+5}}{n^5+5n^2+1}$$

Converge or Diverge?

$$\frac{5n\sqrt{n+5}}{n^5+5n^2+1} \le \frac{5n\sqrt{n+5n}}{n^5+5n^2+1}$$

$$= \frac{5n\sqrt{6}\cdot\sqrt{n}}{n^5+5n^2+1}$$

$$\le \frac{5\sqrt{6}\cdot n^{3/2}}{n^5}$$

$$= \frac{5\sqrt{6}}{n^{7/2}}$$

And $\sum \frac{1}{n^{7/2}}$ converge via $p>1$ test.

Example!

$$\sum_{n=1}^{\infty} \frac{8^n}{3^{2n+1}} - \frac{3}{n(n+1)}$$

sol $-\frac{1}{3}$

- $\frac{8^n}{3^{2n+1}} = \frac{8^n}{3 \cdot 3^{2n}} = \frac{8^n}{3 \cdot (3^2)^n} = \frac{1}{3} \cdot \left(\frac{8}{9}\right)^n$

and $0 \leq \frac{8}{9} < 1$ so

$$\frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{8}{9}} - 1\right)$$

$$= \frac{1}{3}(9-1)$$

$$= \frac{8}{3}$$

- $\frac{3}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{(n)(n+1)}$

$$= \frac{(A+B)n + A}{(n)(n+1)}$$

So $A = 3$, $B = -3$

Telescope

Then $\sum_{n=1}^{\infty} \underbrace{\frac{3}{n}}_{b_n} - \underbrace{\frac{3}{n+1}}_{b_{n+1}} = b_1 - \lim_{n \to \infty} b_n = 3$

Since both $\sum \frac{8^n}{3^{2n+1}}$, $\sum \frac{3}{n(n+1)}$ conv separately,

we know

Algebraic laws.

$$\sum_{n=1}^{\infty} \frac{8^n}{3^{2n+1}} - \frac{3}{n(n+1)} = \sum_{n=1}^{\infty} \frac{8^n}{3^{2n+1}} - \sum_{n=1}^{\infty} \frac{3}{n(n+1)} = \frac{8}{3} - \frac{9}{3} = -\frac{1}{3}$$

Example: (Converge or Diverge)

$$3/5 - \frac{9}{25} + \frac{27}{125} - \frac{81}{625} + \cdots$$

$$= \frac{3}{5} + (-1)^1 \frac{3^2}{5^2} + (-1)^2 \frac{3^3}{5^3} + (-1)^3 \frac{3^4}{5^4} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{n+1}}{5^{n+1}} = \frac{3}{5} \left(\sum_{n=0}^{\infty} \left(\frac{-3}{5}\right)^n \right)$$

When $\left|\frac{-3}{5}\right| < 1 \Rightarrow \sum_{n=0}^{\infty} \left(\frac{-3}{5}\right)^n = \frac{1}{1-\left(\frac{-3}{5}\right)}$

$$= \frac{1}{\frac{8}{5}} = 5/8.$$

So

$$\frac{3}{5} \left(\sum_{n=0}^{\infty} \left(\frac{-3}{5}\right)^n \right) = \frac{3}{5}\left(5/8 \right) = \boxed{3/8} \quad \text{conv.}$$

Example:

$$\sum \frac{3^n}{8^n + n}$$

, direct comparison

Example:

$$\sum \sin\left(\frac{1}{n^2}\right)$$

limit comparison

- $a_n := \sin\left(\frac{1}{n^2}\right)$, $n \geq 0$
- $b_n := \frac{1}{n^2}$, $\sum b_n$ conv ≥ 0.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} =$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\sin\left(\frac{1}{n^2}\right)} \overset{LH}{=} \lim_{n\to\infty} \frac{\cancel{-2n^{-3}}}{\cos\left(\frac{1}{n^2}\right)\cdot \cancel{-2n^{-3}}}$$

$$= \lim_{n\to\infty} \frac{1}{\cos\left(\frac{1}{n^2}\right)}$$

$$= 1 > 0.$$

So sine $\sum \frac{1}{n^2}$ conv, by limit Compare $\sum \sin\left(\frac{1}{n^2}\right)$ conv.