# Power Series

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## Last Time

- Limit Comparison
- Alternating Series

## Today

Power series

- Radius of Conv.
- Interval of Conv.

## Questions

- What is a function ?
- What is a power series?
- Why are power series important?
  - Construct new functions
  - Represent original functions as power series.

# Power Series

## What is a function?

Loosely speaking a function is a rule that takes an input and uniquely assigns an output value.

## What is a power series?

- Given a sequence $(c_n)$, and fixed number $a$, we can construct an expression

$$\sum_{n=0}^{\infty} c_n (z-a)^n .$$

**Central Question**: what values of $z$ are we allowed to plug in?

Then we think $z \mapsto \sum_{n=0}^{\infty} c_n (z-a)^n$ as a function

## Radius of convergence:

- Every power series has a unique $0 \leq R \leq \infty$ s.t $\sum_{n=0}^{\infty} c_n (z-a)^n$
  - converge absolutely when $|z-a| < R$
  - diverge when $|z-a| > R$

This number is called the <u>Radius of converge</u>.

# Three Examples

① $\sum_{n=1}^{\infty} \frac{x^n}{n}$ R.o.c? I.o.c?

$R = \lim_{n\to\infty} \frac{n}{n+1} = 1$ $[-1, 1)$

② $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ $R=1$ I.o.c $[-1, 1]$

③ $\sum_{n=1}^{\infty} n x^n$ R.o.c? I.o.c?

$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{n}{n+1} = 1$ / $(-1, 1)$

④ $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ R.o.c?

$R =$ $(-\infty, \infty)$

* The "Faster the coefficient grow" the bigger the radius of convergence.

We mean $(a_n)$ grows faster than $(b_n)$ if $\left(\frac{a_n}{b_n}\right) \to \infty$ or $\left(\frac{b_n}{a_n}\right) \to 0$.

# Useful Facts

① **Thm**: If the power series $\sum_{n=0}^{\infty} a_n z^n$ converge at some $b \neq 0$, then it converges for all $|z| < b$.

Pf:

Why can't we say at $|z| \leq b$?
— See examples above who I.O.C $[-R, R)$

# The Ratio Test

Given $(a_n)$ sequence, and suppose

$$q = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

If $q<1$, then $\sum_{n=0}^{\infty} a_n$ converge

If $q>1$, then $\sum_{n=0}^{\infty} a_n$ diverge

If $q=1$, inconclu.

Proof: Suppose $q<1$. We need to show $\lim_{k\to\infty} \sum_{n=0}^{k} |a_n|$ converge. Well since $q<1$,

pick $q<s<1$. Then since $\left|\frac{a_{n+1}}{a_n}\right| \to q$

$\exists N>0$ s.t

$$\frac{|a_{N+1}|}{|a_N|} < q+\varepsilon < s$$

$$\Rightarrow \quad |a_{N+1}| < s|a_N|$$

But $\frac{|a_{N+2}|}{|a_{N+1}|} < s$

$\Rightarrow \quad |a_{N+2}| < S|a_{N+1}| < S^2 |a_N|$

In general,

$$|a_{N+k}| < S^k |a_N|$$

fixed const

So $\displaystyle\sum_{n=N}^{\infty} |a_n| < \sum_{n=N}^{\infty} |a_N| S^n$

and since $S<1$, by geometric series and direct comparison,

$$\sum_{n=N}^{\infty} |a_n| \text{ conv}$$

- If $q>1$, let $\varepsilon>0$ s.t $q-\varepsilon>1$,

then $\exists N>0$ s.t

$$\frac{|a_{n+1}|}{|a_n|} > q-\varepsilon, \quad \text{all } n \geq N$$

$\Rightarrow \quad |a_{N+1}| > (q-\varepsilon) \cdot |a_N|$

$\Rightarrow \quad |a_{N+2}| > (q-\varepsilon)|a_{N+1}| > (q-\varepsilon)^2 |a_N|$

So we bound from below by

$$\sum_{n=N}^{\infty} |a_n| > \sum_{n=N}^{\infty} |a_N| \cdot (q-\varepsilon)^n$$

and $q-\varepsilon > 1$, so geometric series diverge. then $\sum_{n=N}^{\infty} |a_n| = \infty$

So not absolute conv.

$\boxed{\xi}$

## Types of Questions

① Find radius of convergence

- use ratio test, root test, Geometric series.
- Ratio Test works sometimes.

## Examples 1 (Finding R.O.C)

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 3^n} \cdot x^n$$

$$C_n = \frac{(-1)^n}{n \cdot 3^n}.$$

$$R = \lim_{n\to\infty} \frac{(n+1)\cancel{3}^{n+1}}{n \cdot \cancel{3^n}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot 3 = 3$$

What happens for $\pm 3$?

- $x = 3$, we get $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv by alt series test.
- $x = -3$, we get divergence since

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div.}$$

**Example 2** (Most important function in Mathematics)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n = \infty.$$

Has radius of convergence all of $\mathbb{R}$.

**Example:**

$$\sum_{n=0}^{\infty} \frac{2n(2n-2)\cdots 4\cdot 2}{n!} (x-2)^n$$

$$\lim_{n\to\infty} \frac{2n\cancel{(2n-2)\cdots 4\cdot 2}}{n!} \cdot \frac{(n+1)!}{(2n+2)\,(2n)\cancel{\cdots 4\cdot 2}}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+2} = 2\lim_{n\to\infty} \frac{n+1}{n+1} = \boxed{2.} \quad \text{RoC}$$

And interval of conv $|x-2| < 2$

$\Rightarrow \quad -2 < x-2 < 2$

$\Rightarrow \quad 0 < x < 4$ What happens at

$I = (0,4)$ 0 and 4?

4. Find the interval of convergence of the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ (solution: $(-1, 1]$)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n (x-4)^n}{n}$ (solution: $[15/4, 17/4)$)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{4^n}$ (solution: $(-2, 2)$)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$ (solution: $(-\infty, \infty)$)

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n! (x-5)^n}{3^n}$ (solution: $x = 5$)

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2x)^n}{n^2+1}$ (solution: $[-1/2, 1/2]$)

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ (solution: $(-3, 1)$)

5. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} x^n$. (solution: 1/27.)

Exercise:

$$\sum \frac{(-1)^{n+1} x^{2n}}{n(\ln(n))^2}$$

- Is this still a power series?
- Yes, a lot of 0 coefficients.

- Center?
- R.O.C?
- Endpoints?
- Interval of Conv?

$$R = \lim_{n\to\infty} \frac{(n+1)(\ln(n+1))^2}{n \cdot \ln(n)^2}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \left(\frac{\ln(n+1)}{\ln(n)}\right)^2 \overset{*}{=} 1.$$

$\overset{*}{\text{well}}$ $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$

So $R = 1$.

Converge at $\pm 1$ by alternating series test.

**Exercise:** $\sum_{n=0}^{\infty} \frac{n^2 (x+4)^n}{2^{3n}}$

Center? $a = -4$

R.O.C?

Interval of conv?

Endpoints?

- $\lim_{n\to\infty} \frac{n^2}{2^{3n}} \cdot \frac{2^{3(n+1)}}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \cdot 2^3 = 8$

So R.O.C is 8:

- we need $|x-a| < R$

$\Leftrightarrow |x-(-4)| < 8$

$\Leftrightarrow -8 < x+4 < 8$

$\Leftrightarrow -12 < x < 4$

So I.O.C $I = (-12, 4)$

- What happens at the endpoints?

- $\sum \frac{n^2 (8)^n}{2^{3n}} = \sum \frac{n^2 8^n}{(8)^n} = \sum n^2 \cdot \left(\frac{8}{8}\right)^n = \infty$

- $\sum \frac{n^2 (-8)^n}{8^n} = \sum (-1)^n n^2 \cdot \left(\frac{8}{8}\right)^n = \infty$, by test for div.

Exercise

$$\sum_{n=0}^{\infty} \frac{n!\; x^n}{100^n}$$

Find the R.O.C.

$$\lim_{n\to\infty} \frac{n!}{100^n} \cdot \frac{100^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{100}{n+1} = 0$$

So R.O.C $R = 0$. Meaning the only $x$ for which the series converge is $0$.

**Exercise 1** : Find radius of conv & Interval of Conv.

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} \cdot x^n$$

$$\lim_{n\to\infty} \left( \frac{(2n)!}{n^{2n}} \cdot \frac{(n+1)^{2n+2}}{(2n+2)!} \right) = \lim_{n\to\infty} (n+1)^2 \cdot \left(\frac{n+1}{n}\right)^{2n} \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+2n+2n+2} \cdot \left(\frac{n+1}{n}\right)^{2n}.$$

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)^{2n} = \lim_{n\to\infty} 2n \cdot \ln\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n}\right)}{1/2n}$$

$$\frac{1 \cdot n - (n+1) \cdot 1}{(n+1)^2} = \frac{-1}{(n+1)^2}$$

$$\overset{L'H}{=} \lim \frac{\frac{n}{n+1} \cdot \frac{-1}{(n+1)^2}}{-\frac{1}{2} \cdot \frac{1}{n^2}}$$

$$\frac{1}{2} n^{-1}$$

$$-\frac{1}{2} n^{-2}$$

$$= \lim 2 \cdot \frac{n^3}{(n+1)^3}$$

$$= 2\left(\lim_{n\to\infty} \frac{n}{n+1}\right)^3$$

$$= 2.$$

So $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n} = e^2 \Rightarrow \boxed{\frac{1}{4} e^2}$

Exerci 3:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{(n+1)^2}$$

$$R = \lim_{n\to\infty} \frac{1}{(n+1)^2} \Big/ \frac{1}{(n+2)^2} = \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^2 = 1$$

So we have convergence of $(-4-1, -4+1)$

$= (-5, -3)$

$x = -5$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-5-4)^n}{n+1}$$

$= \sum_{n=0}^{\infty} \frac{9^n}{n+1}$ , diverge since $\left(\frac{9^n}{n+1}\right) \to \infty$

$x = -3$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-3-4)^n}{n+1} = \sum \frac{7^n}{n+1} = \infty$$

I.O.C. $(-5, -3)$

# Exercise 4

$$\sum_{n=0} \frac{(-2)^n x^{n+1}}{n+1}$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{2^n}{n+1} \cdot \frac{n+2}{2^{n+1}} = \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{1}{2} = \frac{1}{2}$$

Center : $0$ so $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$$\sum_{n=0}^{\infty} \frac{(-2)^n \cdot \left(\frac{1}{2}\right)^{n+1}}{n+1} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-2)^n \cdot \left(\frac{1}{2}\right)^n}{n+1}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$ conv by alt. ser

$$\sum_{n=0}^{\infty} \frac{(-2)^n \cdot \left(\frac{-1}{2}\right)^{n+1}}{n+1} = \frac{-1}{2} \sum_{n=0}^{\infty} \frac{1}{n+1}$$ div

$\left(-\frac{1}{2}, \frac{1}{2}\right]$ I. O. C.

## Exer ③

$$\sum \frac{n!\,(x-1)^n}{(2n)!}$$