Discussion 11

Nov 5, 2024

Power Series Repr.

Last Time

Power series
- Radius of Conv.
- Interval of Conv.

Today

Power Series Representations.

Ideas: (Main ways to obtain power series rep)

① Geometric Series

NB!: Fix a number z,

$$\sum_{n=0}^{\infty} z^n = \begin{cases} \frac{1}{1-z}, & |z|<1 \\ \text{diverge}, & |z| \geq 1 \end{cases}$$

② Termwise differentiation and integration along with geometric series.

Examples we look at:

- $f(x) = \frac{1}{3-x}$ centered at $a=0$.
- $f(x) = \frac{1}{3-x}$ centered at $a=2$
- $f(x) = \frac{1}{3-x}$ centered at $a=7$
- $f(x) = \frac{x^3}{3-x^2}$, $a=0$.
- $f(z) = \frac{1}{(1+z)^3}$, $a=0$.
- $f(x) = \frac{\arctan(\sqrt{x})}{\sqrt{x}}$, $a=0$.
- $\int_0^1 \ln(1+x)\,dx$, approximate with accuracy $< 0.01 = \frac{1}{100}$

Example 1: Find power series representations

a.) $f(x) = \frac{1}{3-x}$, centered at $a=2$.

well $\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}}$

$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$, $\left|\frac{x}{3}\right| < 1$

$= \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$, I.O.C $(-3, 3)$

b.) $f(x) = \frac{1}{3-x}$, centered at $a=2$

$= \frac{1}{1-(x-2)}$

$= \sum_{n=0}^{\infty} (x-2)^n$ $\quad -1 < x-2 < 1 \implies -1 \quad x < 3$

with I.O.C $-1 < x < 3$.

c.) $f(x) = \frac{1}{3-x}$, centered at 7

$= \frac{1}{7-4-x}$

$$= \frac{1}{-4 - x + 7}$$

$$= \frac{1}{-4 - (x-7)}$$

$$= \frac{1}{-4} \cdot \frac{1}{1 - \left(\frac{x-7}{-4}\right)}$$

$$\left|\frac{x-7}{-4}\right| < 1$$

$$= \frac{1}{-4} \cdot \sum_{n=0}^{\infty} \left(\frac{x-7}{-4}\right)^n$$

$$= \frac{-1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (x-7)^n}{4^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot (x-7)^n}{4^{n+1}}$$

Example 2: $f(x) = \dfrac{x^3}{3-x^2}$

$$x^3 \cdot \frac{1}{3-x^2} = \frac{x^3}{3} \cdot \frac{1}{1-\frac{x^2}{3}}$$

$$= \frac{x^3}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{x^2}{3}\right)^n \qquad \left|\frac{x^2}{3}\right| < 1$$

$$= \frac{x^3}{3} \cdot \sum \frac{x^{2n}}{3^n}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+3}}{3^{n+1}}$$

$$x^2 < 3 \Rightarrow -\sqrt{3} < x < \sqrt{3}$$

Differentiating and Integrating Power series.

Let $\sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series with radius of convergence $R>0$. Then

① $$\frac{d}{dx}\sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (z-a)^n$$

$$= \sum_{n=1}^{\infty} n \cdot a_n (z-a)^{n-1}$$

and has radius of convergence R

② $$\int \sum_{n=0}^{\infty} a_n (z-a)^n \, dz = \sum_{n=0}^{\infty} \int a_n (z-a)^n \, dz$$

$$= C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-a)^{n+1}$$

and has radius of convergence R

Example 3:

$$g(x) = \frac{1}{(1-x)^2}$$

$$h(x) = \ln(5-x)$$

Two examples where this idea works!

Example 4 :

$$f(z) = \frac{1}{(1+z)^3} \quad , \quad a = 0 .$$

$$h(z) = \frac{1}{1+z} \quad , \quad |z| < 1$$

$$h'(z) = \frac{d}{dz}(1+z)^{-1} = -(1+z)^{-2}$$

$$h''(z) = \frac{d}{dz} - (1+z)^{-2} = 2(1+z)^{-3} = 2f(z).$$

well $h(z) = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n \cdot z^n$

$$h'(z) = \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot z^{n-1}$$

$$h''(z) = \sum_{n=2}^{\infty} (-1)^n \cdot n \cdot (n-1) \cdot z^{n-2}$$

So $f(z) = \sum_{n=2}^{\infty} \frac{1}{2} \cdot (-1)^n \cdot n \cdot (n-1) \cdot z^{n-2}$, I.o.c $(-1,1)$

Example

$$f(x) = \frac{\arctan(\sqrt{x})}{\sqrt{x}}, \quad a = 0.$$

- $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, I.O.C $[-1, 1]$

- $\arctan(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (\sqrt{x})^{2n} \cdot \sqrt{x}}{2n+1}$, allow when
  - $x \geq 0$
  - $-1 \leq x \leq 1$

$$= \sum \frac{(-1)^n \cdot x^n \cdot \sqrt{x}}{2n+1}$$

- Both conditions must hold.

- $\frac{\arctan(\sqrt{x})}{\sqrt{x}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2n+1}$, allowed when
  - $x > 0$
  - $-1 \leq x \leq 1$

So $x > 0$ and $-1 \leq x \leq 1 \Rightarrow$ I.O.C as

$$\boxed{0 < x \leq 1}$$

Error Estimation: (Alternating Series Version)

Recall that if given a series $\sum_{n=0}^{\infty} (-1)^n a_n$

s.t ① $a_n \geq 0$

② (a_n) decreasing

③ $\lim_{n \to \infty}$

By AST we have converge. We can also estimate

error. If $S_N := \sum_{n=0}^{N} (-1)^n a_n$, then the

error at step N is:

$$E_N = \left| \sum_{n=0}^{\infty} (-1)^n a_n - S_N \right| \leq a_{N+1}$$

Why? Well

$$\sum_{n=0}^{\infty} (-1)^n a_n - \sum_{n=0}^{N} (-1)^n a_n$$

$$= \sum_{n=N+1}^{\infty} (-1)^n a_n .$$

If $N+1$ is even,

$$a_{N+1} \underbrace{- a_{N+2} + a_{N+3}}_{\geq 0} \underbrace{- a_{N+4} + a_{N+5}}_{\geq 0} - \dots$$

since $a_{N+2} \geq a_{N+3}$

So $$\sum_{n=N+1}^{\infty} (-1)^n a_n \leq a_{N+1}$$

If $N+1$ odd

$$-a_{N+1} + a_{N+2} \underbrace{- a_{N+3} + a_{N+4}}_{\leq 0} \underbrace{- a_{N+5} + \dots}_{\leq 0}$$

$$\leq \underbrace{-a_{N+1} + a_{N+2}}_{\leq 0} < a_{N+1}$$

So $$\sum_{n=N+1}^{\infty} (-1)^n a_n \leq a_{N+1}$$

If $N+1$ even

$$a_{N+1} - a_{N+2} \underbrace{+ a_{N+3} - a_{N+4}}_{\geq 0} \underbrace{+ a_{N+5} - \dots}_{\geq}$$

$$\geq \underbrace{a_{N+1} - a_{N+2}}_{\geq 0} \geq -a_{N+1}$$

If $N+1$ odd

$$-a_{N+1} \underbrace{+ a_{N+2} - a_{N+3}}_{\geq 0} \underbrace{+ a_{N+4} - a_{N+5}}_{\geq 0} + \dots$$

$$\geq -a_{N+1}$$

Thus

$$-a_{N+1} \leq \sum_{n=0}^{\infty} (-1)^{n} a_n - S_N \leq a_{N+1} .$$

Approximating Integrals :

① $\int_0^1 \ln(1+x)\,dx$ (see below)

② $\int_0^1 \frac{x}{1+x^5}\,dx$ (exercise)

Example: How much term is need to approximate $\int_0^1 \ln(1+x)\,dx$ with accuracy $< 0.01 = \frac{1}{100}$?

Recall that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad IOC\ (-1, 1]$$

So $\int_0^1 \ln(1+x)\,dx$

$$= \sum_{n=1}^{\infty} \int_0^1 (-1)^{n+1} \frac{x^n}{n}$$

$$= \sum_{n=1}^{\infty} \left((-1)^{n+1} \cdot \frac{x^{n+1}}{n\cdot(n+1)} \Bigg|_0^1 \right)$$

$$= \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n\cdot(n+1)}}_{\text{alternating series.}}$$

Want $a_{N+1} < \frac{1}{100}$

$$\Rightarrow \frac{1}{(N+1)(N+2)} < \frac{1}{100}$$

Find smallest integer that works.

$$\Rightarrow N^2 + 3N + 2 > 100$$

$$\Rightarrow N^2 + 3N - 98 > 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4 \cdot (-98)}}{2}.$$

$$= \frac{-3 \pm \sqrt{401}}{2}$$

$N > 0 \Leftarrow$

$$\frac{-3 + \sqrt{401}}{2} > \frac{-3 + \sqrt{400}}{2}$$

$$= \frac{-3 + 20}{2}$$

$$= \frac{17}{2} = 8.5$$

- There are only two N's that solve

$N^2 + 3N + 2 = 100$

↳ one is negative and the

other just bigger than 8.5.

- 8 wont work since the

$8^2 + 3 \cdot 8 + 2 < 100$.

- But 9 will since

$9^2 + 3 \cdot 9 + 9 > 8.5^2 + 3 \cdot 8 + 8 > 100$.

- So $a_{9+1} < \frac{1}{100}$

$$\Rightarrow \left| \int_0^1 \ln(1+x)\,dx - \sum_{n=1}^{9} (-1)^{n+1} \cdot \frac{1}{n \cdot (n+1)} \right|$$

$< a_{9+1} < \frac{1}{100}$.

9 is the smallest term s.t error $< \frac{1}{100}$.

Example: Approximate the following integral:

$$\int_0^1 \frac{x}{1+x^5}\,dx$$

Q: Find upper bound on error when using 11th non-neg term.

① Find power series expansion for $\frac{x}{1+x^5}$

② termwise integration

③ Apply Taylor Remainder or Alternating series estimation.

① well we use geometric series formula,

$$x\frac{1}{1+x^5} = x\frac{1}{1-(-x^5)}$$

condition $|-x^5|<1$

$$= x\sum_{n=0}^{\infty}(-x^5)^n$$

$$= x\sum_{n=0}^{\infty}(-1)^n x^{5n}$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{5n+1}.$$

② Termwise integration:

$$\int_0^1 \frac{x}{1+x^5}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{5n+1}\,dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{5n+1}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+2}}{5n+2} \Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{5n+2} .$$

And we can use A.S. error approximation.

I.e for example

$$E_{10} = \left| \int_0^1 \frac{x}{1+x^5} dx - \sum_{n=0}^{10} (-1)^n \frac{1}{5n+2} \right|$$

$$\leq \frac{1}{5(10+1)+2} .$$

a_{n+1}

Some additional Results (taken from Understand analysis)

- This first result tells us that if a power series converge at some point x_0, then it converges absolutely on $(-|x_0|, |x_0|)$, i.e. on an entire interval with radius $|x_0|$.

Thm (6.5.1): Given a power series $\sum_{n=0}^{\infty} a_n x^n$, if $\sum_{n=0}^{\infty} a_n x^n$ converges at x_0, then

$\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $|x| < |x_0|$.

(pf: compare with convergent Geometric series)

This next result is used along with the fact that a power series converge absolute inside its radius of convergence

Thm (6.5.2)

Power series converge uniformly on compact subsets contained in there ROC.

More precisely, given $\sum_{n=0}^{\infty} a_n x^n$,

if $\sum_{n=0}^{\infty} a_n x^n$ converge absolutely at x_0,

then $\sum_{n=0}^{k} a_n x^n \to \sum_{n=0}^{\infty} a_n x^n$ uniformly on

$[-c, c]$ where $c := |x_0|$.

(Pf: use absolute convergence to show that the power series is uniform Cauchy).

pf (6.5.1)

Given $\lim_{k\to\infty} \sum_{n=0}^{k} a_n x_0^n$ converge, let $|x| < |x_0|$.

- Since we have convergence at x_0, we know $a_n x_0^n \to 0$. Thus $(a_n x_0^n)$ bounded close to zero. That is $\exists M>0$, and $N_0>0$

s.t.

$$-M < a_n x_0^n < M, \text{ all } n \geq N_0$$

$$\Rightarrow |a_n| < M \cdot \frac{1}{|x_0|^n}, \text{ all } n \geq N.$$

- By Δ-inequality then

we have $\sum_{n=K}^{N} |a_n x|^n \leq \sum_{n=K}^{N} M \left|\frac{x}{x_0}\right|^n$, all $N \geq N_0$

Since $\left|\frac{x}{x_0}\right| < 1$, we know $\sum_{n=K}^{\infty} M \cdot \left|\frac{x}{x_0}\right|^n$ convergent geometric series, and hence by direct comparison $\sum_{n=K}^{\infty} |a_n x|^n$. Thus $\sum_{n=0}^{\infty} |a_n x|^n$ converge

$\blacksquare$

Pf (6-8.2)

- It suffices to show uniform Caucy.

Since $\sum_{n=0}^{\infty} a_n x^n$ converge absolutt at x_0,

we know $\left(\sum_{n=0}^{k} |a_n| \cdot |x_0|^n \right)_{k=0}^{\infty}$ is a Caucy seq.

Let $\varepsilon > 0$. Choose $N > 0$ s.t

$$\left| \sum_{n=0}^{k} |a_n| \cdot |x_0|^n - \sum_{n=0}^{m} |a_n| \cdot |x_0|^n \right| < \varepsilon$$

$$\quad\quad = \sum_{n=m+1}^{k} |a_n| \cdot |x_0|^n$$

Thus $\sum_{n=m+1}^{k} |a_n| \cdot |x_0|^n < \varepsilon$, all $k > m \geq N$.

Let $k > m \geq N$, and observe

Then $\left| \sum_{n=0}^{k} a_n x^n - \sum_{n=0}^{m} a_n x^m \right|$

$$\leq \sum_{n=m+1}^{k} |a_n| \cdot |x|^n \leq \sum_{n=m+1}^{k} |a_n| \cdot |x_0|^n < \varepsilon, \quad \text{for all } |x| \leq |x_0|$$

and x's indepedt of N. Hence we showed uniform Caucy $\Rightarrow$ uniform converge

$\boxed{S}$