Discussion 12

Nov 12, 2024

Taylor Series, Taylor's Remainder Thm

Last Time:

- Power series Representations
  - Using Geometric series, term-wise integration and differentiation.

Today:

- Taylor Series
- Taylor's Remainder Theorem.

Introduction:

Using the geometric series, along with termwise differentiation, and integration of power series on its domain of convergence we obtain power series representations of:

- $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ with ZOC $[-1, 1]$
- $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ with ZOC $(-1, 1]$

And many more functions.

Lagrange's Remainder Thm (Understanding Analysis)

(page 200)

Thm

Let f be a $N+1$ times differentiable function on $(-R,R)$, define $a_n := \frac{f^{(n)}(0)}{n!}$, and

- $S_N := a_0 + a_1 x + \dots + a_N x^N$, for each $0 \leq n \leq N$
- $E_N := f - S_N$ (error function)

Then given $x \neq 0 \in (-R,R)$ $\exists$ $|c| < |x|$ s.t

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

- That is for a fixed $x \neq 0 \in (-R,R)$, we know the error of approximating $f(x)$ with $S_N(x)$ is $\frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$

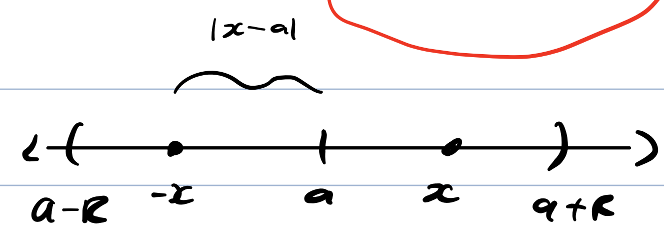
⊛ So the error depends on x and c.

Taylor's Remainder Theorem

- Given a function f infinitely differentiable on some $(a-R, a+R)$, fixed center a, radius $R>0$.
- let
  - $a_n := \dfrac{f^{(n)}(0)}{n!}$, all $n = 0, 1, 2, \dots$
  - $S_n(x) := a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$
  - $E_n(x) = f(x) - S_n(x)$ (Error / Remainder)

Given a fixed $x \neq 0$ in $(a-R, a+R)$, if there exists $M \geq 0$ s.t.

$$|f^{(N+1)}(c)| \leq M \quad \text{for all } |c-a| < |x-a|$$



then

$$|E_N(y)| \leq \frac{M|y-a|^{N+1}}{(N+1)!} \quad \text{all}$$

$|y-a| < |x-a|$

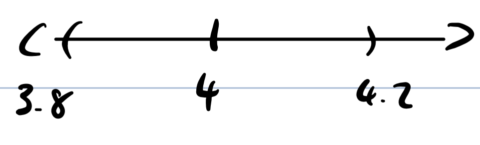
Example: Given $f(x) = \sqrt{x}$, center $a = 4$.

Find the maximum error of using the second degree Taylor series to approx $\sqrt{4.2}$.

n	Derivative	value at $a=4$
0	$x^{\frac{1}{2}}$	$f(4) = 2$
1	$\frac{1}{2} x^{-\frac{1}{2}}$	$f'(4) = 1/4$
2	$\frac{-1}{4} x^{-\frac{3}{2}}$	$f''(4) = \frac{-1}{4} \cdot \frac{1}{8}$
3	$3/8\, x^{-5/2}$	$f'''(4) = 3/8 \cdot \frac{1}{32}$

Fix $x_0 = 4.2$

Error $|R_2(x_0)| = |f(x_0) - S_2(x_0)|$



To use Taylor's remainder thm we need to:

- Find $M \geq 0$ s.t

$$|f^{(2+1)}(c)| \leq M \quad \text{all } c \text{ between } 4 \text{ and } 4.2.$$

• well $f^{(3)}(c) = 3/8 \cdot \frac{1}{(\sqrt{c})^5}$

$\leq 3/8 \cdot \frac{1}{(\sqrt{4})^5}$, all $4 < c < 4.2$

Since $\frac{1}{(\sqrt{c})^5}$ increases as c goes from 4.2 to 4.

So $|f^{(3)}(c)| \leq \frac{3}{8} \cdot \frac{1}{2^5} = \frac{3}{2^8} = \boxed{\frac{3}{256}}$ choice for M.

$$|E_N(y)| \leq \frac{M|y-a|^{N+1}}{(N+1)!}$$

So $|R_2(4.2)| \leq \frac{\frac{3}{256} \cdot |4.2-4|^3}{3!}$

$$= \frac{3 \cdot (0.2)^3}{256 \cdot 3!}$$

Example (Application of Taylor Series)

Evaluate

$$\lim_{x \to 0} \frac{\cos(x^4) - 1 + \frac{1}{2}x^8}{x^{16}}$$

Recall

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{on } (-\infty, \infty)$$

So $$\cos(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n}}{2n!}$$

coeff at $n=2$ is $\frac{1}{4!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n}}{2n!} = 1 - \frac{1}{2}x^8 + \ldots$$

So $$\frac{\cos(x^4) - 1 + \frac{1}{2}x^8}{x^{16}} = \frac{\sum_{n=2}^{\infty} \frac{(-1)^n x^{8n}}{(2n)!}}{x^{16}}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n x^{8n-16}}{(2n)!}$$

$0^0 = 1$

And $$\lim_{x \to 0} \sum_{n=2}^{\infty} \frac{(-1)^n x^{8n-16}}{(2n)!} = \sum_{n=2}^{\infty} \frac{(-1)^n (0)^{8n-16}}{(2n)!} = \boxed{\frac{1}{(2 \cdot 2)!}}$$

coeff.

power series are continuous.

Example: $f(x) = e^{x^2}$

Find $f^{(100)}(0)$.

Recall $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

So $e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$, i.e. $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$.

Also $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{n!} x^k$.

Set $\underset{k=100}{100} = 2n$, $\Rightarrow$ $\underset{n=50}{n=50}$.

So $$\frac{f^{(100)}(0)}{100!} = \frac{1}{50!}$$

$$\Rightarrow \quad f^{(100)}(0) = \frac{100!}{50!}$$

Example: Find the sum of the series:

$$\boxed{f\left(\tfrac{1}{2}\right) = \ln\left|\overset{+1}{}\tfrac{3}{2}\right|} \quad \checkmark$$

Or $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{1}{2}\right)^n = \ln\left(1+\frac{1}{2}\right)$$

Long way

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} \; ?$$

Idea is to find the function it represents.

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{1}{2}\right)^n$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot x^n \quad , \text{ think } f\left(\tfrac{1}{2}\right)$$

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}$$

$$= \sum_{n=1}^{\infty} (-x)^{n-1}$$

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$$

$$\sum_{n=1}^{\infty} y^n = \frac{1}{1-y} - 1$$

$$= \frac{1-(1-y)}{1-y}$$

$$= \frac{y}{1-y}$$

$$= \sum_{n=0}^{\infty} (-x)^n \qquad |x| < 1$$

$$= \frac{1}{1+x}$$

So $f'(x) = \frac{1}{1+x}$

$$\int \frac{1}{1+x}\,dx = \ln|1+x| + c$$

So $f(x) = \ln|1+x| + c$

$f(0) = 0 \Rightarrow \ln|1+0| + c = 0 \Rightarrow c = 0$

So $f(x) = \ln|1+x|$

Example 2 : $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln 2)^n}{n!}$

well $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\displaystyle\Rightarrow \sum_{n=1}^{\infty} \frac{(\ln 2)^n}{n!} = e^{(\ln 2)} - 1 = 2-1 = 1$

Example 3 : Evaluate

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$ (sol $\frac{\sqrt{2}}{2}$)

$\displaystyle= \sum_{n=0}^{\infty} \frac{(-1)}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1} = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Example:

Problem 9:

Taylor's Remainder Estimation Theorem

Given a function f, with Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n \,, \quad \text{center } a.$$

For a fixed number x,

if $|f^{N+1}(c)| \le M$ for all numbers c between x and a, then

N^{th} remainder / Error

$$R_N(x) = f(x) - \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n}_{T_N \text{ - } N^{th} \text{ degree Taylor poly.}}$$

Satisfy $$|R_N(x)| \le \frac{M\,|x-a|^{N+1}}{(N+1)!}$$

Problem 9:

Given: $f(x) = \sqrt{x}$, center $a = 4$.

Find the Maximum error of using the second degree Taylor Series to approx $\sqrt{4.2}$.

n	Derivative	value at $a=4$
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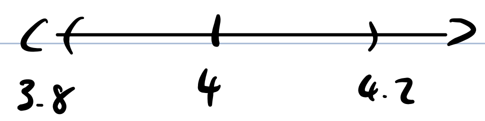
Fix $x_0 = 4.2$

Error $|R_2(x_0)| = |f(x_0) - S_2(x_0)|$



To use Taylor's remainder thm we need to:

- Find $M \geq 0$ s.t

$\left| f^{(2+1)}(c) \right| \leq M$ all c between 4 and 4.2.

- well $f^{(3)}(c) = 3/8 \cdot \dfrac{1}{(\sqrt{c})^5}$

$\leq 3/8 \cdot \dfrac{1}{(\sqrt{4})^5}$, all $4 < c < 4.2$

Since $\dfrac{1}{(\sqrt{c})^5}$ increases as c goes from 4.2 to 4.

So $\left| f^{(3)}(c) \right| \leq \dfrac{3}{8} \cdot \dfrac{1}{2^5} = \dfrac{3}{2^8} = \boxed{\dfrac{3}{256}}$ choice for M.

$$|E_N(y)| \leq \frac{M |y-a|^{N+1}}{(N+1)!}$$

So $$|R_2(4.2)| \leq \frac{\frac{3}{256} \cdot |4.2 - 4|^3}{3!}$$

$$= \frac{3 \cdot (0.2)^3}{256 \cdot 3!}$$

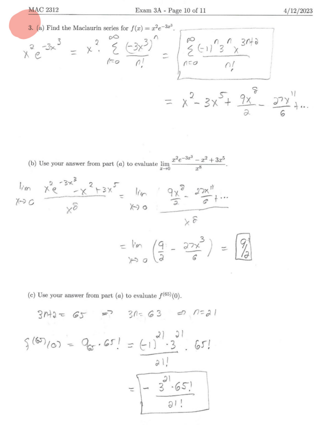
Example (Spring 2023, FR Q3)

a) Find Taylor series of $x^2 \cdot e^{-3x^3}$ about 0.

b) Use your answer to evaluate

$$\lim_{x \to 0} \frac{x^2 e^{-3x^3} - x^2 + 3x^5}{x^8}$$

c) Use part a) to evaluate $f^{(65)}(0)$.



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3. (a) Find the Maclaurin series for $f(x) = x^2 e^{-3x^3}$.

$$x^2 e^{-3x^3} = x^2 \cdot \sum_{n=0}^{\infty} \frac{(-3x^3)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{3n+2}}{n!}}$$

$$= x^2 - 3x^5 + \frac{9x^8}{2} - \frac{27x^{11}}{6} + \dots$$

(b) Use your answer from part (a) to evaluate $\lim_{x \to 0} \frac{x^2 e^{-3x^3} - x^2 + 3x^5}{x^8}$.

$$\lim_{x \to 0} \frac{x^2 e^{-3x^3} - x^2 + 3x^5}{x^8} = \lim_{x \to 0} \frac{\frac{9x^8}{2} - \frac{27x^{11}}{6} + \dots}{x^8}$$

$$= \lim_{x \to 0} \left(\frac{9}{2} - \frac{27x^3}{6} \right) = \boxed{9/2}$$

(c) Use your answer from part (a) to evaluate $f^{(65)}(0)$.

$3n+2 = 65 \Rightarrow 3n = 63 \Rightarrow n = 21$

$$f^{(65)}(0) = a_{65} \cdot 65! = \frac{(-1)^{21} \cdot 3^{21}}{21!} \cdot 65!$$

$$= \boxed{-\frac{3^{21} \cdot 65!}{21!}}$$

a) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-3x^3} = \sum_{n=0}^{\infty} \frac{1}{n!} (-3x^3)^n$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \cdot (-1)^n x^{3n}$$

$$x^2 \cdot e^{-3x^3} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3^n}{n!} \right) x^{3n+2}$$

$$= x^2 - 3x^5$$

b) $$\frac{x^2 e^{-3x^3} - x^2 + 3x^5}{x^8} = \frac{\sum_{n=2}^{\infty} (-1)^n \frac{3^n}{n!} x^{3n+2}}{x^8}$$

$$= \sum_{n=2}^{\infty} (-1) \frac{3^n}{n!} x^{3n-6}$$

$$\lim_{x \to 0} \sum_{n=2}^{\infty} (-1) \frac{3^n}{n!} x^{3n-6} = \frac{3^2}{2!} = \frac{9}{2}$$

c.) Find

$f^{(65)}(0)$

Common Maclaurin Series

$f(x)$	Maclaurin Series	Interval of Convergence
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\ln(1-x)$	$\displaystyle-\sum_{n=1}^{\infty} \frac{x^n}{n}$	$[-1, 1)$
$\ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$	$(-1, 1]$
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\cos x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n\frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\sin x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\arctan x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n\frac{x^{2n+1}}{2n+1}$	$[-1, 1]$