Discussion 16

Dec 3, 2024

Last Time

- Exam 3 Review

Today

- Arc length
- Parametric equations
- Polar Co-ordinates
- Graphing Polar Co-ordinates

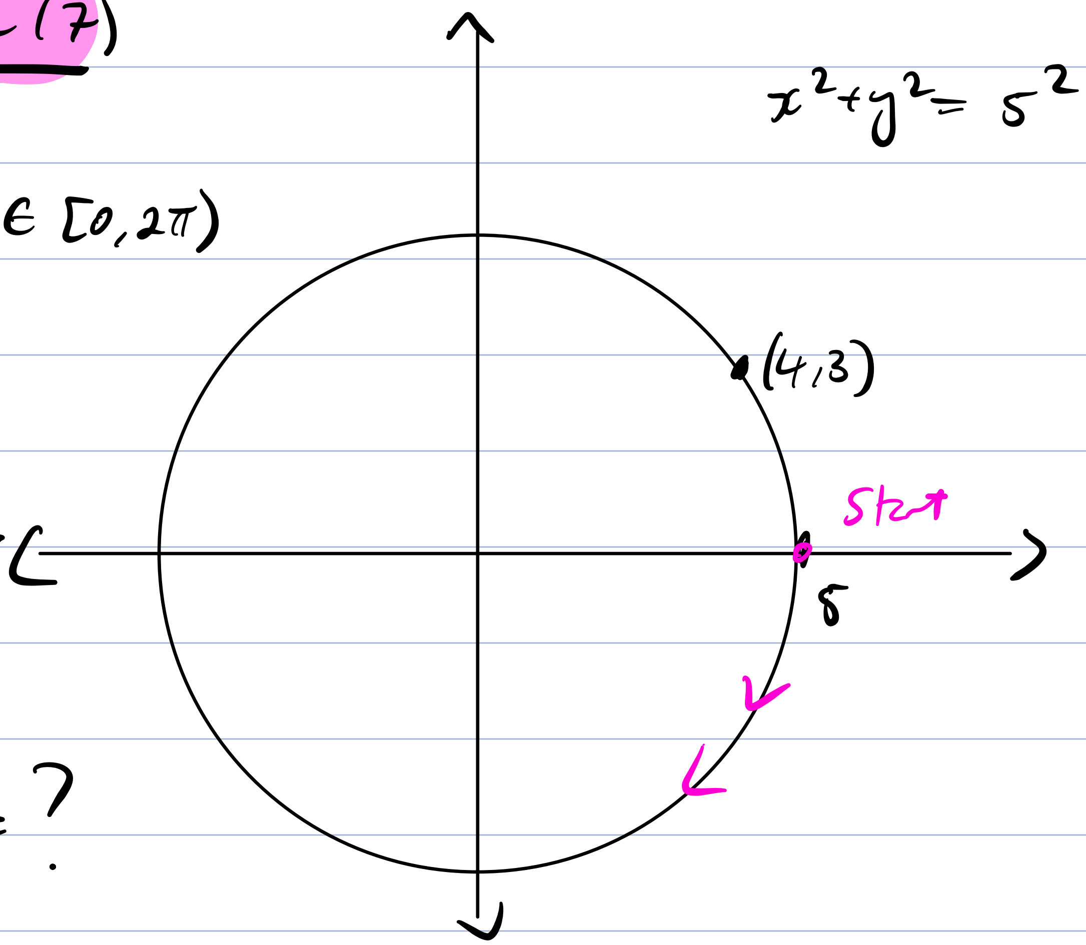
Remark: Graphing equations in polar co-ordinates is needed to solve a lot of these problems.

Lecture 32 Extra Problems :

Extra Problems:

1. Find the equation of the tangent line to the curve $x(t) = \sec(t), y(t) = \tan(t)$ at $t = \pi/3$. (solution: $x(t) = t + 2, y(t) = \frac{2t}{\sqrt{3}} + \sqrt{3}$)
2. Consider the curve C with the parametric equations $x(t) = \cos t + t \sin t$, $y(t) = \sin t - t \cos t$.
   (a) Find all values of t where C has a horizontal tangent line. (solution: $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$)
   (b) Find all values of t where C has a vertical tangent line. (solution: $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$)
3. Find the length of the parametric curve with equations $x(t) = \arcsin t$, $y(t) = \ln\sqrt{1 - t^2}$ from $t = 0$ to $t = 1/2$. (solution: $\frac{1}{2}\ln 3$)
4. Find $\dfrac{d^2y}{dx^2}$ for the curve $x(t) = \sec(t), y(t) = \tan(t)$ at $t = \pi/3$. (solution: $-\frac{1}{3\sqrt{3}}$)
5. Find the open intervals on which the parametric curve $x(t) = 3t^2$, $y(t) = t^3 - t$ is concave upward and concave downward. (solution: CU: $(0, \infty)$, CD: $(-\infty, 0)$)
6. Find the arclength of the curve $x(t) = e^t \cos(t), y(t) = e^t \sin(t)$ from $t = 0$ to $t = \pi/2$. (solution: $\sqrt{2}(e^{\pi/2} - 1)$)
7. ✓ Suppose a particle moves clockwise along the circle $x^2 + y^2 = 25$. If the particle's velocity in the x direction is 8 at the point $(4, 3)$, what is its velocity in the y direction at that point? Hint: The circle can be parametrized clockwise by $x(t) = 5\sin(at)$, $y(t) = 5\cos(at)$ where a is a constant. (solution: $-32/3$) (see Below)

Example (7)



let $t_0 \in [0, 2\pi)$

s.t

$x(t_0) = 4$

$y(t_0) = 3.$

Find

$\left.\frac{dy}{dt}\right|_{t_0} = ?$

- $x(t) = 5\cos(at) \Rightarrow \frac{dx}{dt} = -5a\sin(at)$
- $y(t) = 5\sin(at) \Rightarrow \frac{dy}{dt} = 5a\cos(at)$

Use $\left.\frac{dx}{dt}\right|_{(4,3)} = 8$ to find a.

So $-5a\sin(at_0) = 8.$

At point $(4,3)$ so $y(t_0) = 3$

$\Rightarrow 5\sin(at_0) = 3$

$\Rightarrow -a(5\sin(t_0)) = 8$

$\Rightarrow -a(3) = 8 \Rightarrow a = -8/3$

Then $\frac{dy}{dt}\Big|_{t_0} = 5a\cos(at_0) = ?$

well $x(t_0) = 4 \Rightarrow 5\cos(at_0) = 4$

$$\Rightarrow \frac{dy}{dt}\Big|_{t_0} = a \cdot 4 = -\frac{8}{3}\cdot 4 = \boxed{-\frac{32}{4}}$$

$\boxed{3}$

Remark: Could also solve using implicit diff

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

& plug in $(4,3)$, and solve $\frac{dy}{dt}\Big|_{t_0}$.

① write parametric eq into x and y.

Example: $x(t) = \cos(t) - 1$, $y(t) = \sec t + \cos(t)$

$$= \frac{1}{\cos(t)} + \cos(t)$$

- $\sin^2(t) + \cos^2(t) = 1$
- $\tan^2(t) + 1 = \sec^2(t)$

$$(x+1)y = \cos(t) \cdot \left(\frac{1}{\cos(t)} + \cos t \right) = 1 + \cos^2(t)$$

$$(x+1)^2 = \cos^2(t).$$

So $(x+1)y - (x+1)^2 = 1 + \cos^2(t) - \cos^2(t)$

$$= 1$$

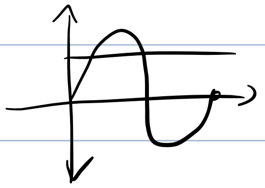
$$\Leftarrow (x+1)y = 1 + (x+1)^2$$

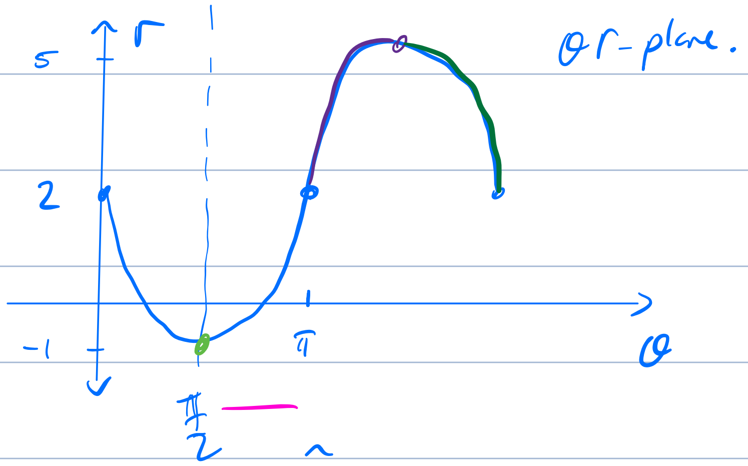
$$= \quad y = \frac{(x+1)^2 + 1}{x+1}$$

Example 3: (Adding Const)

$r = 2 - 3\sin(\theta)$

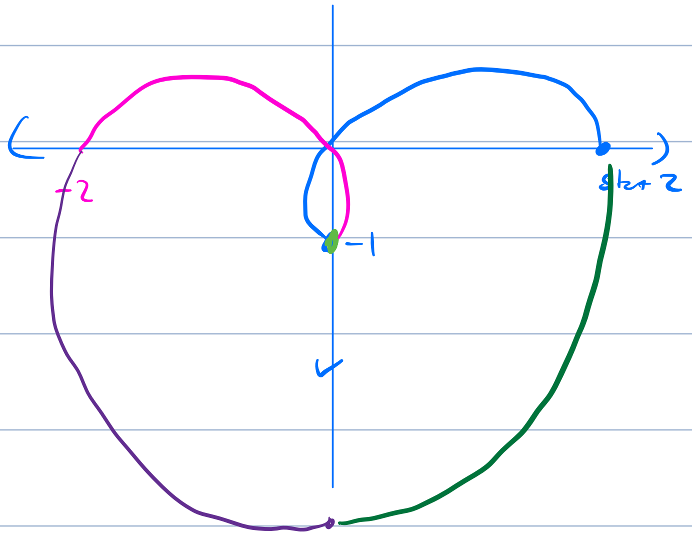
zero: $\sin(\theta) = \frac{2}{3}$





Tip:

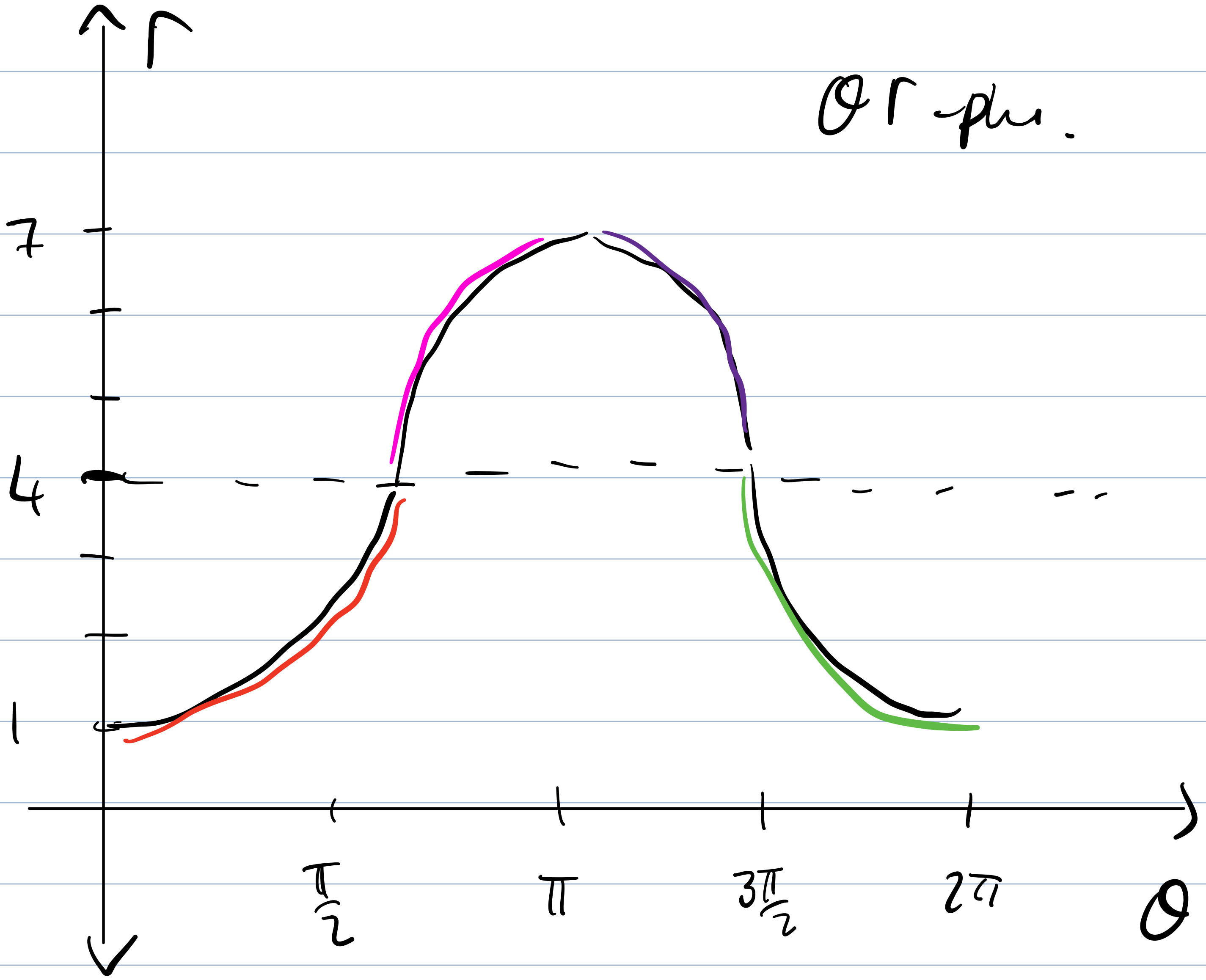
each ╭ ╮ ╰ ╯ is a circle whe θ.



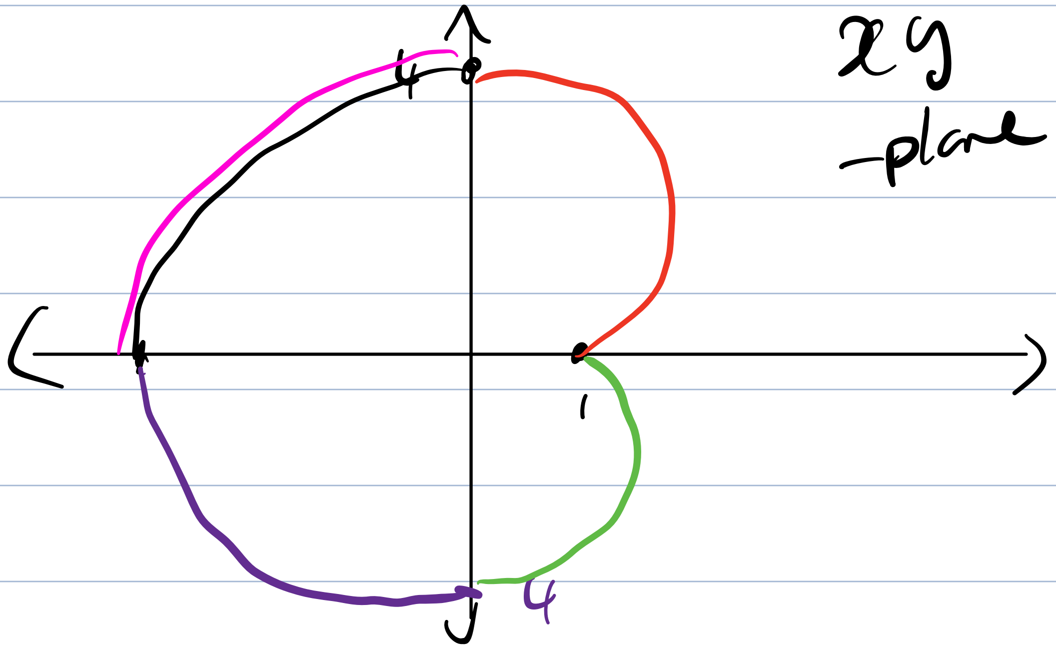
Example:

$r = 4 - 3\cos\theta$

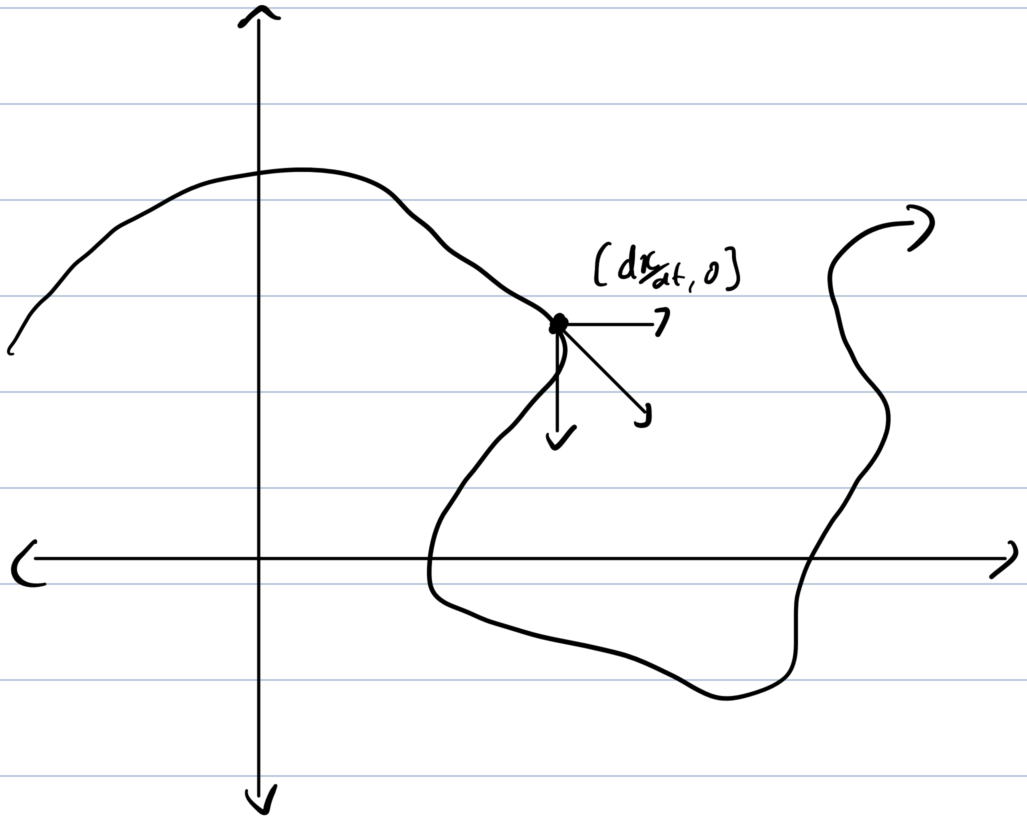
θr-plane.



xy-plane

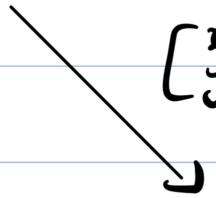


Lecture 32 (calculus with Parametric equations)



$$\begin{bmatrix} \frac{dx}{dt} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{dy}{dt} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$

well



$\begin{bmatrix} x \\ y \end{bmatrix}$, Then the slope of this vector is $\frac{y}{x}$

- $\frac{dy}{dx}$ represent the amount of change in the y-direction wrt 1-unit → in x direction

Lecture 33 : Polar Area

In this lecture we will learn how to find the area within and between polar curves.

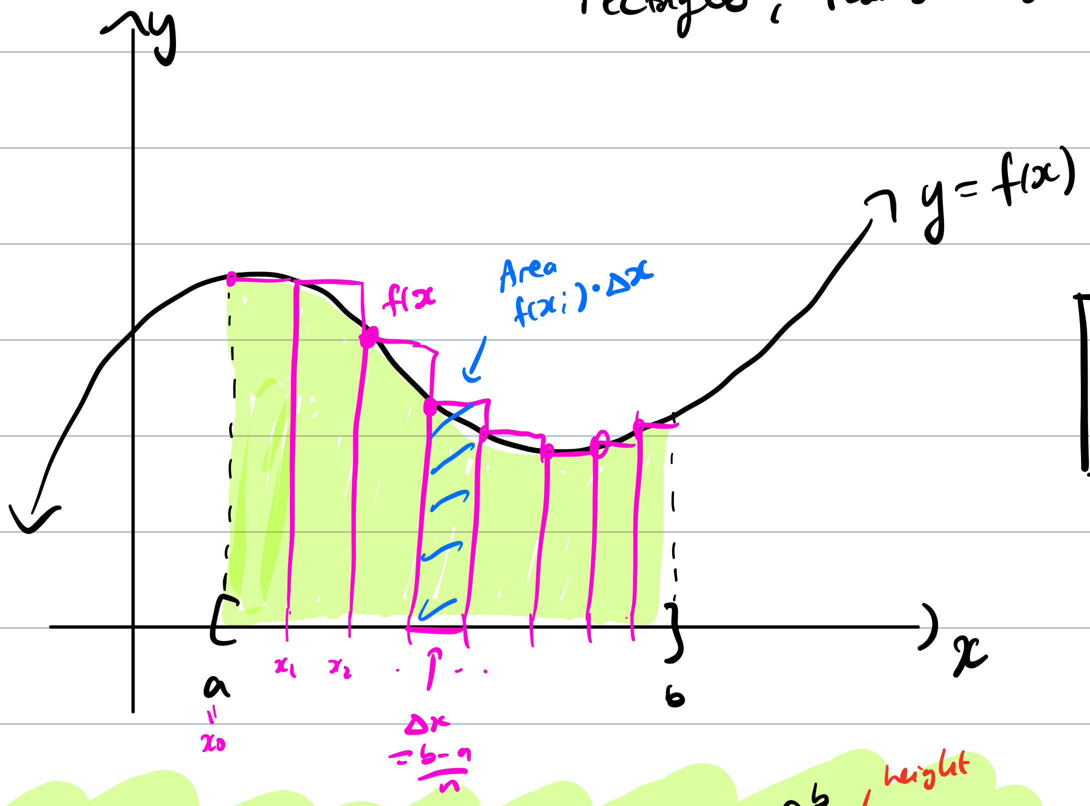
Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of $r = f(\theta)$ between the lines $\theta = a$ and $\theta = b$ is

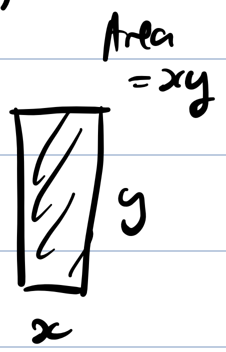
$$A = \frac{1}{2}\int_a^b [f(\theta)]^2\, d\theta.$$

Where does this formula come from?

Rectangular Integrals:

Adding up small rectangles, that's why $f(x_i^*)\Delta x$

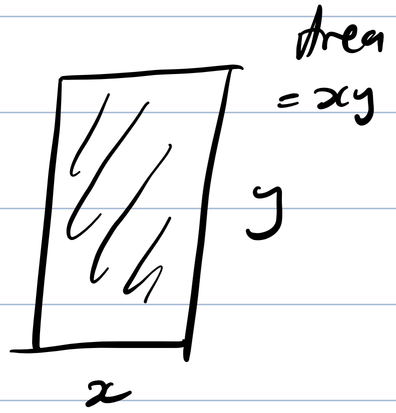




$$\text{Area} = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\cdot\Delta x = \int_a^b \overset{\text{height}}{f(x)}\,\underset{\text{width}}{dx}$$

Integrals With Polar Co-ordinates :

- We sum up sectors of circles instead of rectangles.



Area of sector

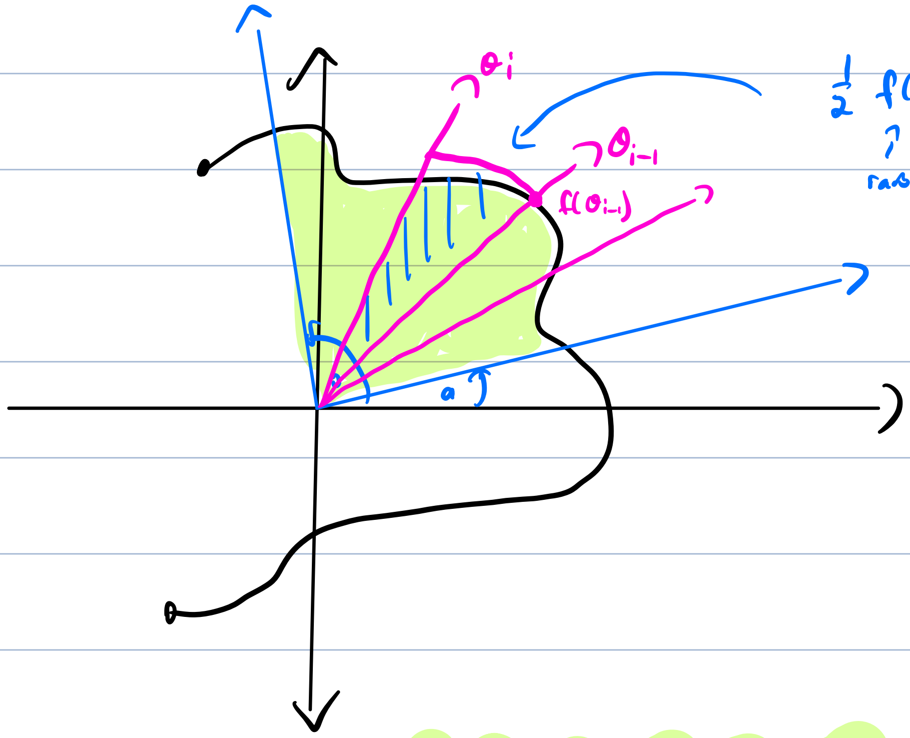
$$= \pi r^2 \cdot \left(\frac{\theta}{2\pi}\right) = \frac{1}{2} r^2 \theta$$

↗ Full circle

↑ proportion of circle

Now $[a,b]$ represents angle.

$r = f(\theta)$



$\frac{1}{2} f(\theta_{i-1})^2 \cdot \Delta\theta$,

↗ rad

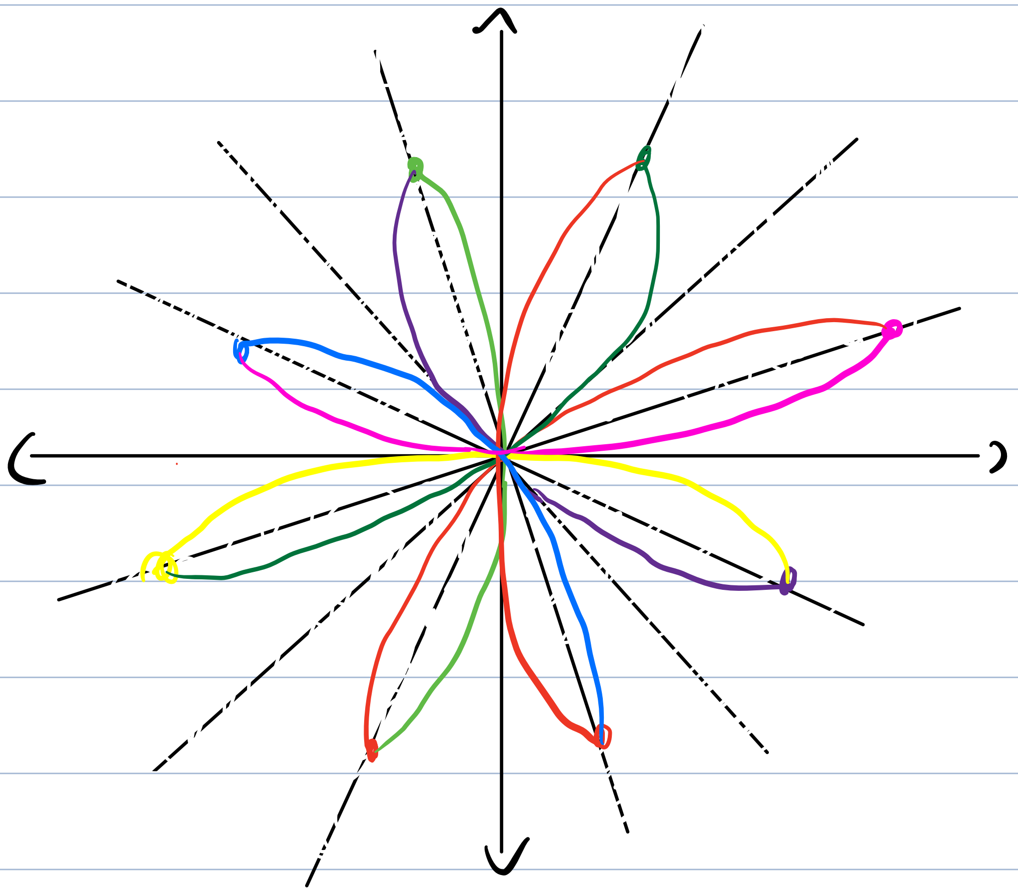
$\Delta\theta = \frac{b-a}{n}$

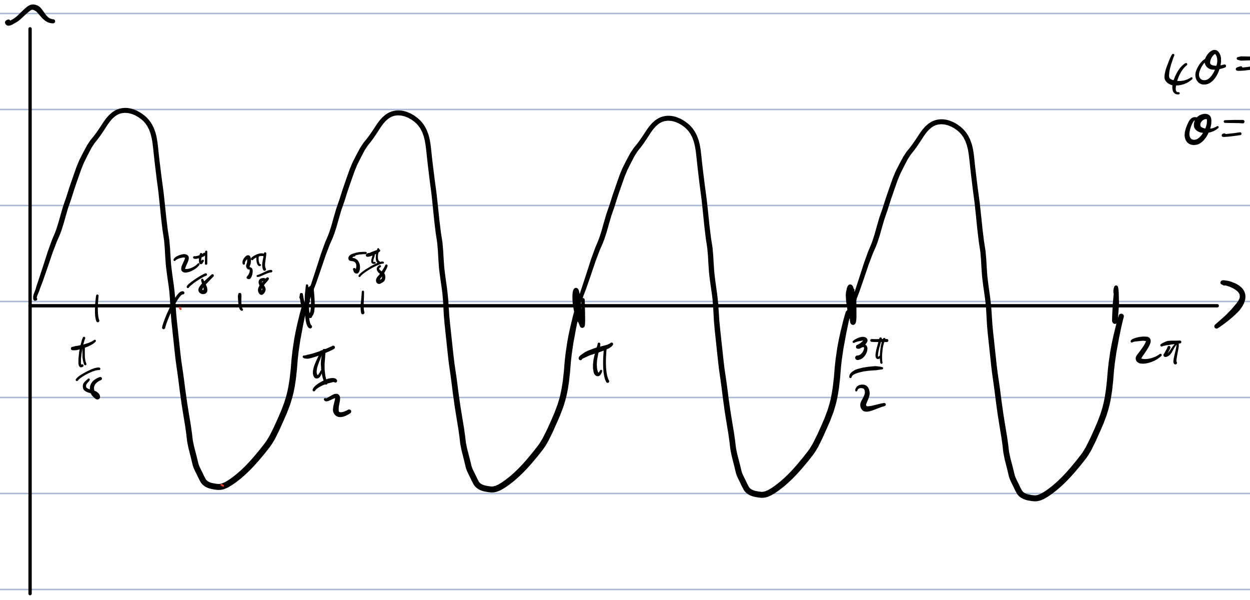
$= \theta_i - \theta_{i-1}$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2} f(\theta_{i-1})^2 \Delta\theta = \frac{1}{2}\int_a^b f(\theta)^2 \, d\theta.$$

Example 1: Find the are enclosed by

$r = \sin(4\theta)$





$4\theta = 2\pi$

$\theta = \frac{\pi}{2}$

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} \sin^2(\theta)\, d\theta$$

$$\cos^2(\theta) = \frac{1}{2} + \frac{\cos(2\theta)}{2}$$

$$= \frac{1}{2}\int 1-\cos^2(\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1-\left(\frac{1}{2}+\frac{\cos(2\theta)}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{2} - \frac{\cos(2\theta)}{2}\, d\theta$$

$$\frac{1}{2}\sin(2\theta)$$

$$= \cos(2\theta)\cdot 2\cdot \frac{1}{2}$$

$$= \frac{1}{4}\int_0^{2\pi} 1-\cos(2\theta)\, d\theta$$

$$= \frac{1}{4}\cdot 2\pi - \frac{1}{4}\int_0^{2\pi} \cos(2\theta)\, d\theta$$

$$= \frac{2\pi}{4} - \frac{1}{4}\cdot\frac{1}{2}\sin(2\theta)\Big|_0^{2\pi}$$

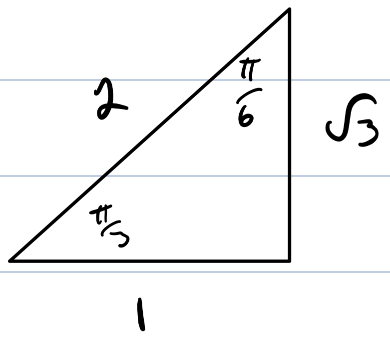
$$= \frac{\pi}{2}.$$

Example 2 :

Find area enclosed by curve $r = 2\sec\theta$ ad rays $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{3}$

$$\text{AREA} = \frac{1}{2}\int_{\pi/6}^{\pi/3} (2\sec\theta)^2 \, d\theta$$

$$= \frac{4}{2}\int_{\pi/6}^{\pi/3} \sec^2\theta \, d\theta$$



$$= 2 \tan(\theta) \Big|_{\pi/6}^{\pi/3}$$

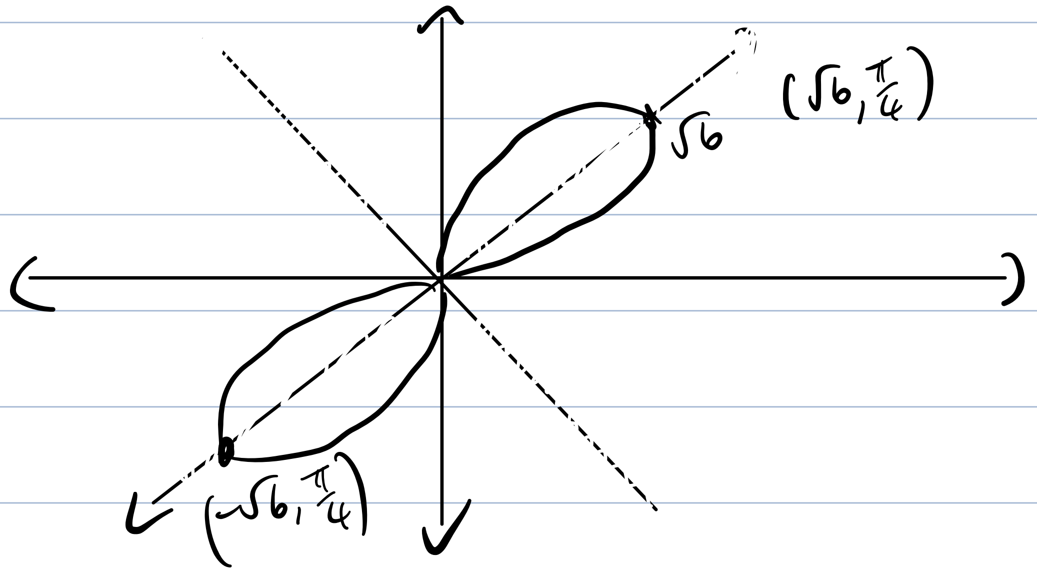
$$= 2 \left[\tan(\pi/3) - \tan(\pi/6)\right)$$

$$= 2\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$$

$$= 2\left(\frac{3-1}{\sqrt{3}}\right)$$

$$= \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

Example: $r^2 = 6\sin(2\theta)$



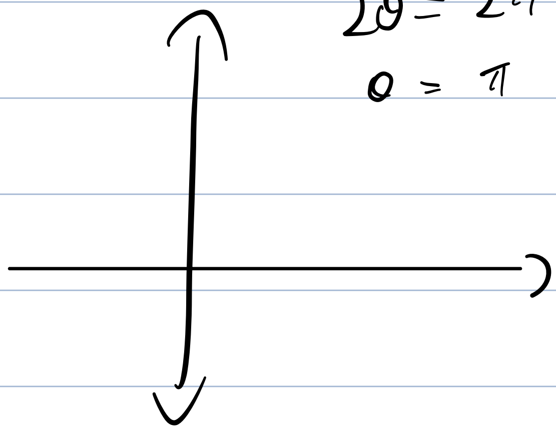
Zeroes: $6\sin(2\theta) = 0$ iff $\sin(2\theta) = 0$ iff

$2\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$

$\iff \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{4\pi}{2}$

$2\theta = 2\pi$

$\theta = \pi$



Max: $2\theta = \frac{\pi}{2}, \frac{5\pi}{2} \iff \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$r^2 = 6\sin(2\theta)$

Only defined for $0 \leq \theta \leq \pi/2$

$\frac{-1}{2}\cos(2\theta)$

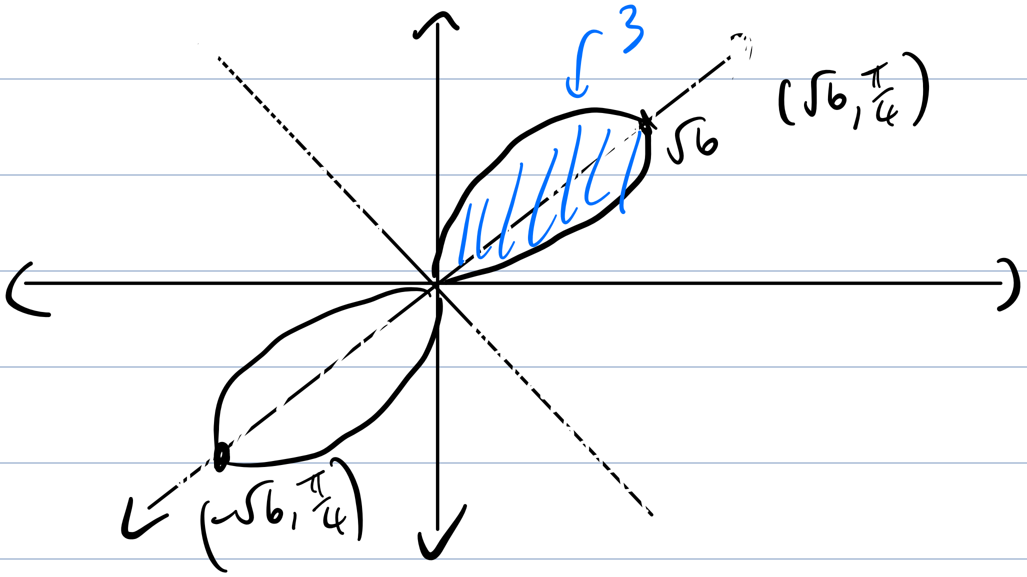
$= \frac{-1}{2}(-\sin(2\theta)) \cdot 2$

$$\frac{\text{Area}}{2} = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\theta = \frac{6}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta) \, d\theta$$

$$= 3 \cdot \frac{-1}{2}\cos(2\theta) \Big|_0^{\frac{\pi}{2}}$$

$$= -3/2\left(\cos(\pi) - \cos(0)\right) = 3$$

Now



Area $= 2 \cdot 3 = 6$

- $r^2 = 6\sin(2\theta)$ defined for

$-\sqrt{6} \le r \le \sqrt{6}$, $0 \le \theta \le \frac{\pi}{2}$

Summary and Extra Exercises.

Arc length:

Given $f:[a,b]\to\mathbb{R}$ continuous. Then the length of the curve $y=f(x)$ is

$$L=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

Arc length (Parametric version)

Given parametric equation

$$x,y:[a,b]\to\mathbb{R}\quad,\quad t\mapsto(x(t),y(t))$$

we get

$$L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\quad dt$$

Example: Sketch $x(t) = \sin^2(t)$, $y(t) = 1+\cos(t)$

$\sin^2(t) + \cos^2(t) = 1.$

$y(t) - 1 = \cos(t)$

$(y(t)-1)^2 = \cos^2(t)$

So $x(t) + (y(t)-1)^2 = 1$

$\Rightarrow x(t) + y(t)^2 - 2y(t) + 1 = 1$

Example: Write as parametric equation.

$$x(t) = 3\cos(3t) + 6$$
$$y(t) = 3\sin(3t) - 7$$

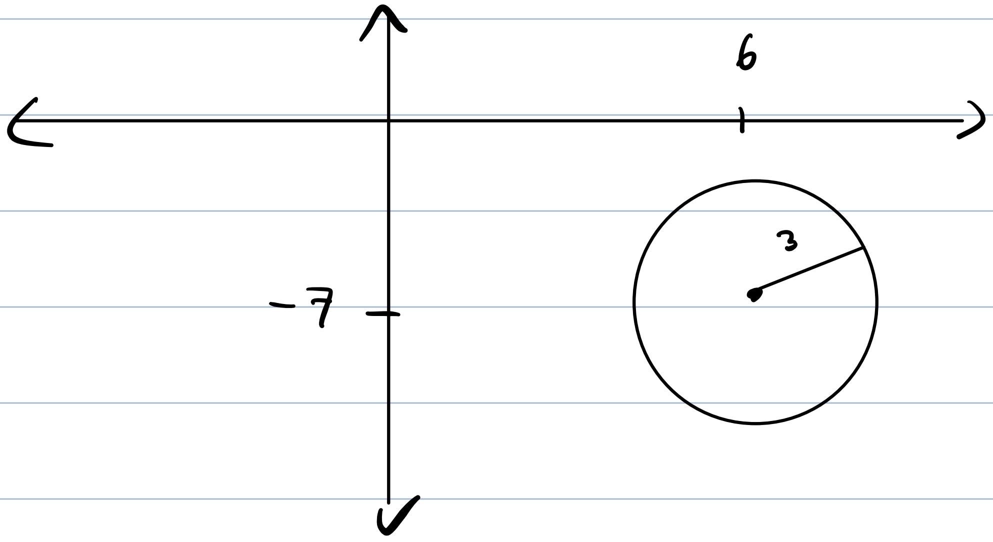
$$\Rightarrow \quad \frac{x-6}{3} = \cos(3t)$$

$$\frac{y+7}{3} = \sin(3t)$$

Since $\cos^2(3t) + \sin^2(3t) = 1$

$$\left(\frac{x-6}{3}\right)^2 + \left(\frac{y+7}{3}\right)^2 = 1$$

$$(x-6)^2 + (y+7)^2 = 9$$



Polar Co-ordinates

- $x^2+y^2 = r^2$
- $x = r\cos(\theta)$
- $y = r\sin(\theta)$

$(x,y) \longleftrightarrow (r,\theta)$

$r \geq 0,$

$0 \leq \theta < 2\pi$

① Find rectangular cordinats.

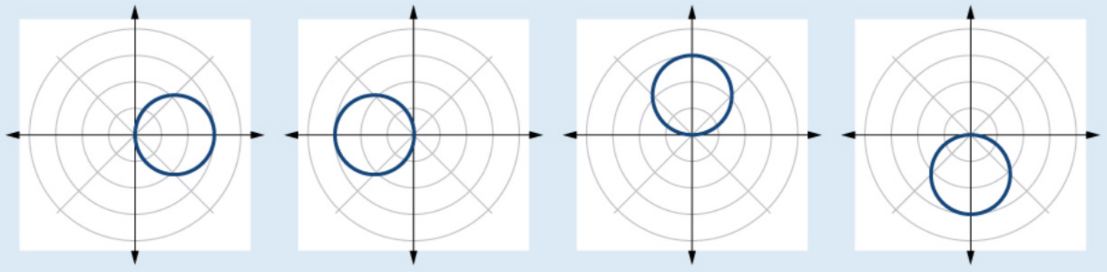
a.) $(6, \frac{\pi}{6})$ so $x = 6\cos(\frac{\pi}{6})$

$y = 6\sin(\pi/6)$

② Find polar co-ordinats:

FORMULAS FOR THE EQUATION OF A CIRCLE

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, wherea a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure $8.4.5$ shows the graphs of these four circles.

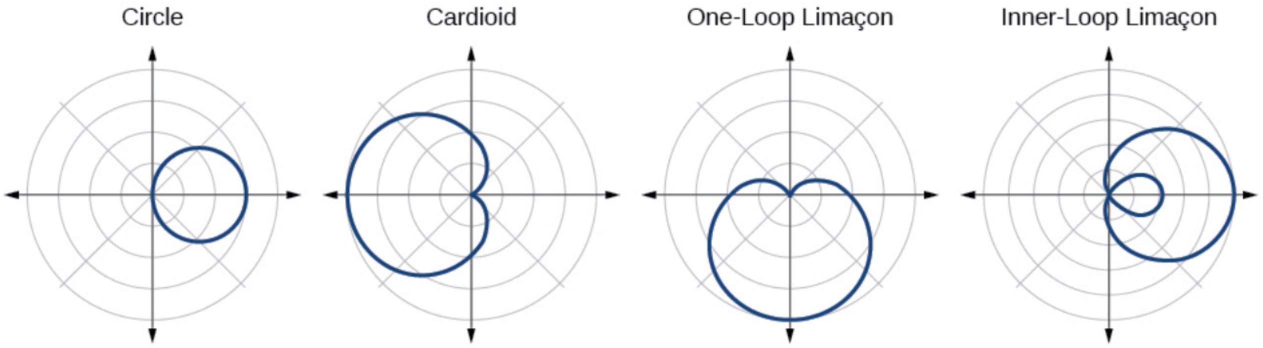


$r = a\cos\theta$, $a > 0$ (a) $r = a\cos\theta$, $a < 0$ (b) $r = a\sin\theta$, $a > 0$ (c) $r = a\sin\theta$, $a < 0$ (d)

Figure $8.4.5$

Summary of Curves

We have explored a number of seemingly complex polar curves in this section. Figure $8.4.24$ and Figure $8.4.25$ summarize the graphs and equations f each of these curves.



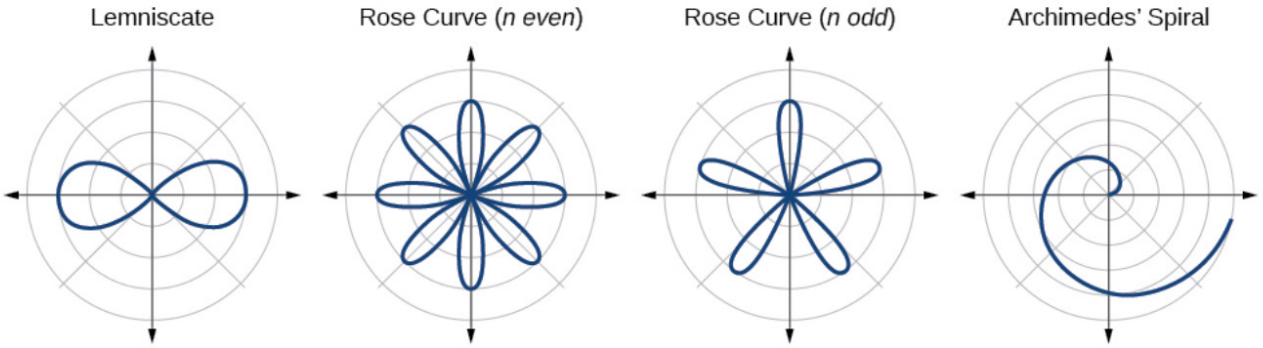
Circle: $r = a\sin\theta$, $r = a\cos\theta$ (a)

Cardioid: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$, $a > 0$, $b > 0$, $a/b = 1$ (b)

One-Loop Limaçon: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$, $a > 0$, $b > 0$, $1 < a/b < 2$ (c)

Inner-Loop Limaçon: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$, $a > 0$, $b > 0$, $a < b$ (d)

Figure $8.4.24$



Lemniscate: $r^2 = a^2\cos 2\theta$, $r^2 = a^2\sin 2\theta$, $a \neq 0$ (a)

Rose Curve (*n even*): $r = a\cos n\theta$, $r = a\sin n\theta$, n even, $2n$ petals (b)

Rose Curve (*n odd*): $r = a\cos n\theta$, $r = a\sin n\theta$, n odd, n petals (c)

Archimedes' Spiral: $r = \theta$, $\theta \geq 0$ (d)

Figure $8.4.25$

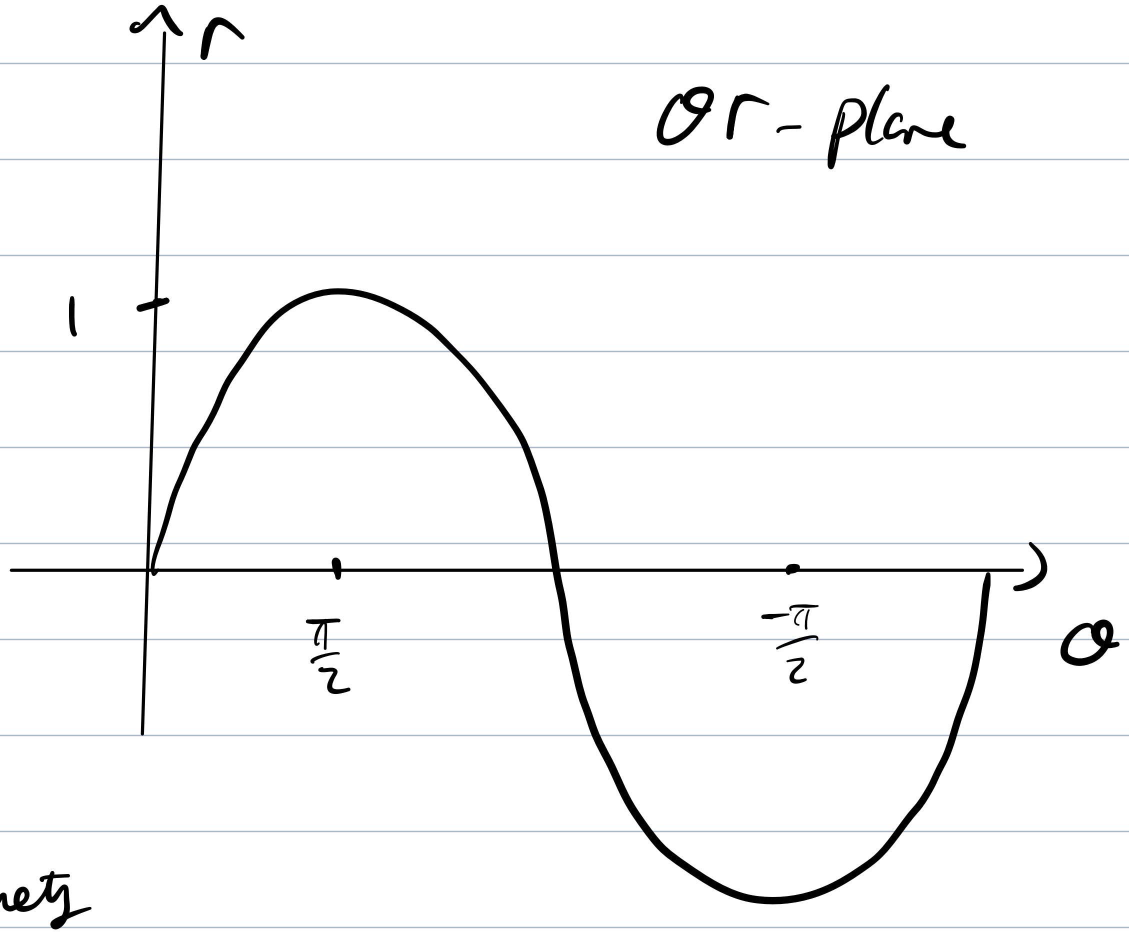
Example: Sketch $r = \sin\theta$.

① Zeros: $\sin(\theta) = 0$, $\theta = \pi k$, $k \in \mathbb{Z}$

② Initial & terminal Points:

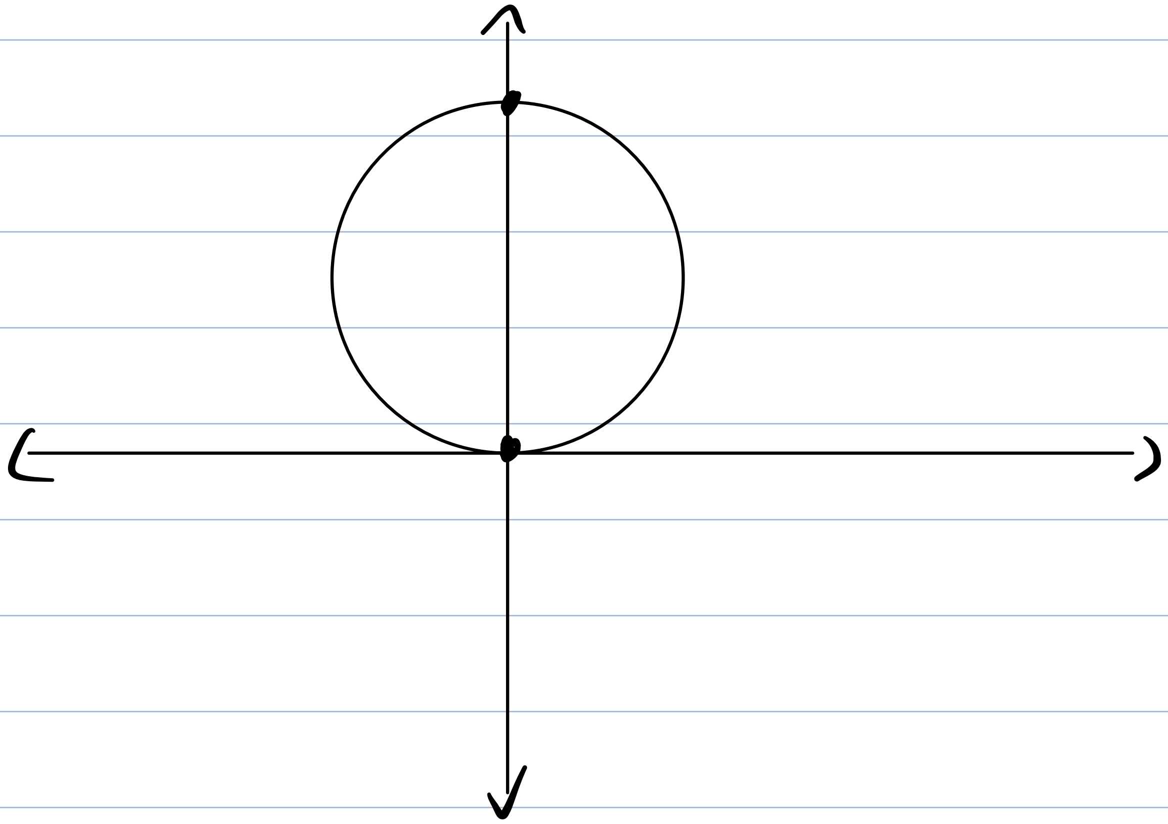
- $\sin(0) = 0$, $\sin(2\pi) = 0$

③ Graph $r = f(\theta)$ as a Cartesian Curve



④ Symmetry

- (r, θ) vs $(-r, -\theta)$



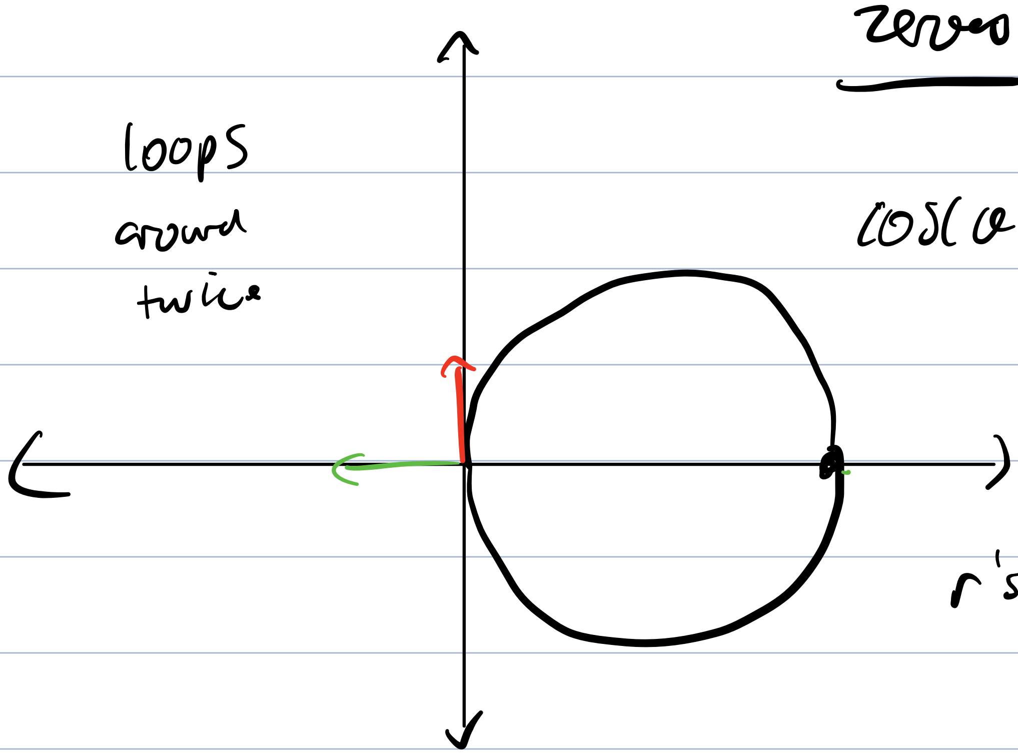
Example 2 :

$r = \cos(\theta)$

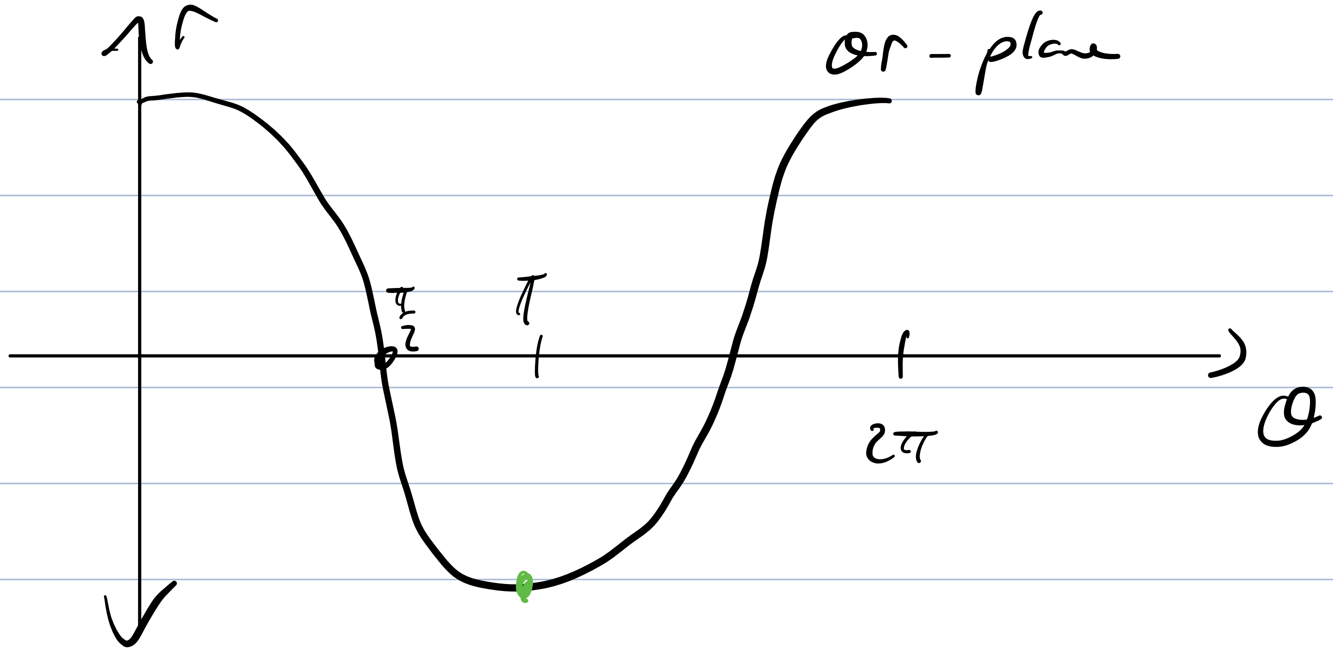
zeros :

$\cos(\theta) = \frac{\pi}{2} \pm \pi k$

loops around twice



r's negah.



r
θr - plane
$\frac{\pi}{2}$
π
2π
θ

Review / Overall

- Work
- New stuff
- Friday
- Volume, surface of revolution.