Mac 2312 Dec 6, 2024

Final Review

Final:

- Exam 1 material (20%)
- Exam 2 material (20%)
- Exam 3 material (20%)
- Post Exam 3 (40%)

Outline:

- Course Overview
- Common Types of questions
- Final Exams
  - Spring 2024 Final
  - Fall 2023 Final.
- Selected Topics

Course Outline

A • Integration

I – Integration Techniques } Exam 1

II – Application of Integration

Exam 2

B • Sequences and Series

I – Infinite Series

II – Power Series } Exam 3

C • Curves in $\mathbb{R}^2$

I – Parametric Curves

II – Calculus with Polar Co-ordinates.

Common Types of Questions

1. Evaluate an integral.
2. Determine if an infinite series converge or diverge (with justification)
3. Find the I.O.C convergence of a power series
4. Find the power series repr. of a function.

Final Exam

Spring 2024

Question 1 : C.I

What is the length of curve $y = 2\sec(t)$, $t = 0$ to $t = \frac{\pi}{4}$.

- Arc length

$$L = \int_a^b \sqrt{1 + f'(t)^2}\, dt.$$

well $f(t) = 2\sec(t)$, $f' = 2\sec(t)\tan(t)$

- So $L = \int_0^{\frac{\pi}{4}} \sqrt{1 + 4\sec^2(t)\tan^2(t)}\, dt$

Sol (A)

Question 2 : A.I

$$\int_0^1 x^2 e^x \, dx$$

- IBP
- Apply Twice.
- $\int u\,dv = uv - \int v\,du$

$u = x^2$, $v = e^x$

$du = 2x\,dx$, $dv = e^x dx$

- $\int_0^1 x^2 e^x \, dx = \underbrace{x^2 e^x \Big|_0^1}_{= e} - \int_0^1 e^x \, 2x \, dx$

$u_1 = 2x$, $v_1 = e^x$

$du_1 = 2dx$, $dv_1 = e^x dx$

- $\int_0^1 x^2 e^x \, dx = e - \left(2x e^x \Big|_0^1 - \int_0^1 2e^x \, dx \right)$

$= e - \left(2e - 2e^x \Big|_0^1 \right)$

$= e - (2e - (2e - 2))$

$= e - (2e - 2e + 2)$

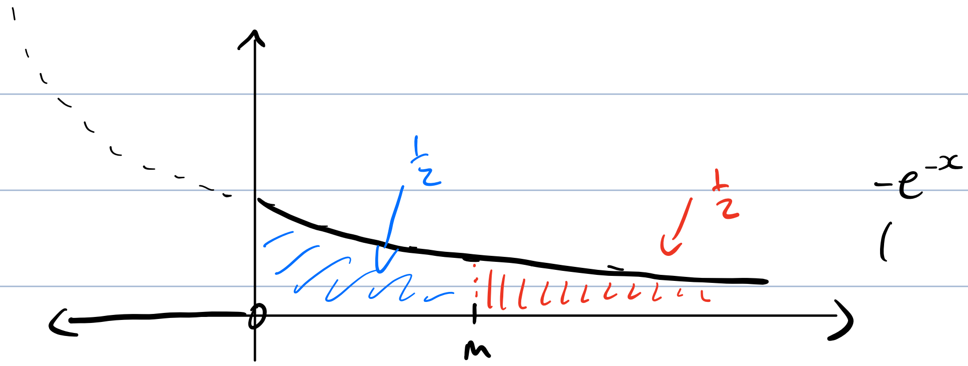
$= e - 2$

Sol D

Question 3 : A.II Probability

Given a probability density function

$$f(x) = \begin{cases} e^{-x} , & x \geq 0 \\ 0 , & \end{cases}$$



Find m s.t.

$$\int_m^{\infty} f(x)\,dx = \frac{1}{2} \quad \text{or} \quad \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$$

$m \geq 0$

well $\displaystyle\int_{-\infty}^{m} f(x)\,dx = \int_0^m e^{-x}\,dx = -e^{-x}\Big|_0^m$

$$= -(e^{-m} - 1)$$

$$= 1 - e^{-m}$$

$$= 1 - \frac{1}{e^m}.$$

So $1 - \frac{1}{e^m} = \frac{1}{2} \Rightarrow \frac{1}{e^m} = \frac{1}{2}$

$$\Rightarrow e^m = 2$$

$$\Rightarrow m = \ln(2)$$

Sol B

Question 4 (B.II)

Find R.O.C of Pow Series.

what is the R.O.C of

$$\sum_{n=1}^{\infty} \frac{n^2 (x+2)^n}{2^{3n}} \; ?$$

Recall the ratio test:

- Given a series $\sum a_n$, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ conv.

Find cond on x.

- So fix some x, and consider

$$a_n = \frac{n^2 (x+2)^n}{2^{3n}}.$$

$3n+3$

- $$\lim_{n\to\infty} \left| \frac{(n+1)^2 (x+2)^{n+1}}{\cancel{2}^{3(n+1)}} \cdot \frac{\cancel{2}^{3n}}{n^2 (x+2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{2^3} \cdot \left(\frac{n+1}{n}\right)^2 \cdot (x+2) \right|$$

$$= \frac{|x+2|}{8} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{|x+2|}{8}.$$

- Now back track, if we assumed x was chosen s.t $\frac{|x+2|}{8} < 1$, then the ratio test tells us that the infinite series for that fixed x conv.

- So $|x+2| < 8 \Rightarrow -8 < x+2 < 8$

$\Rightarrow \underline{-10} < x < \underline{6}$

endpoints.

- Radius of conv is

8

- Sol A

Question 5: C II

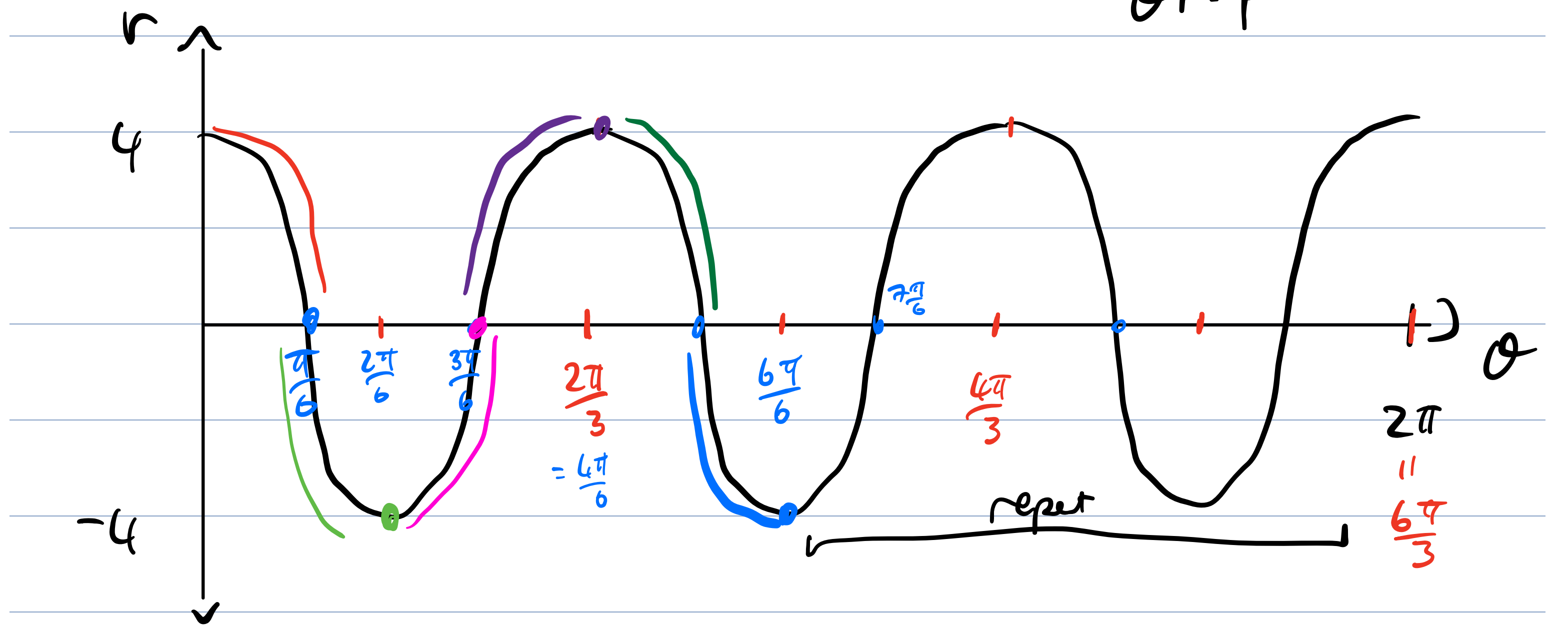
$3\theta = 2\pi$

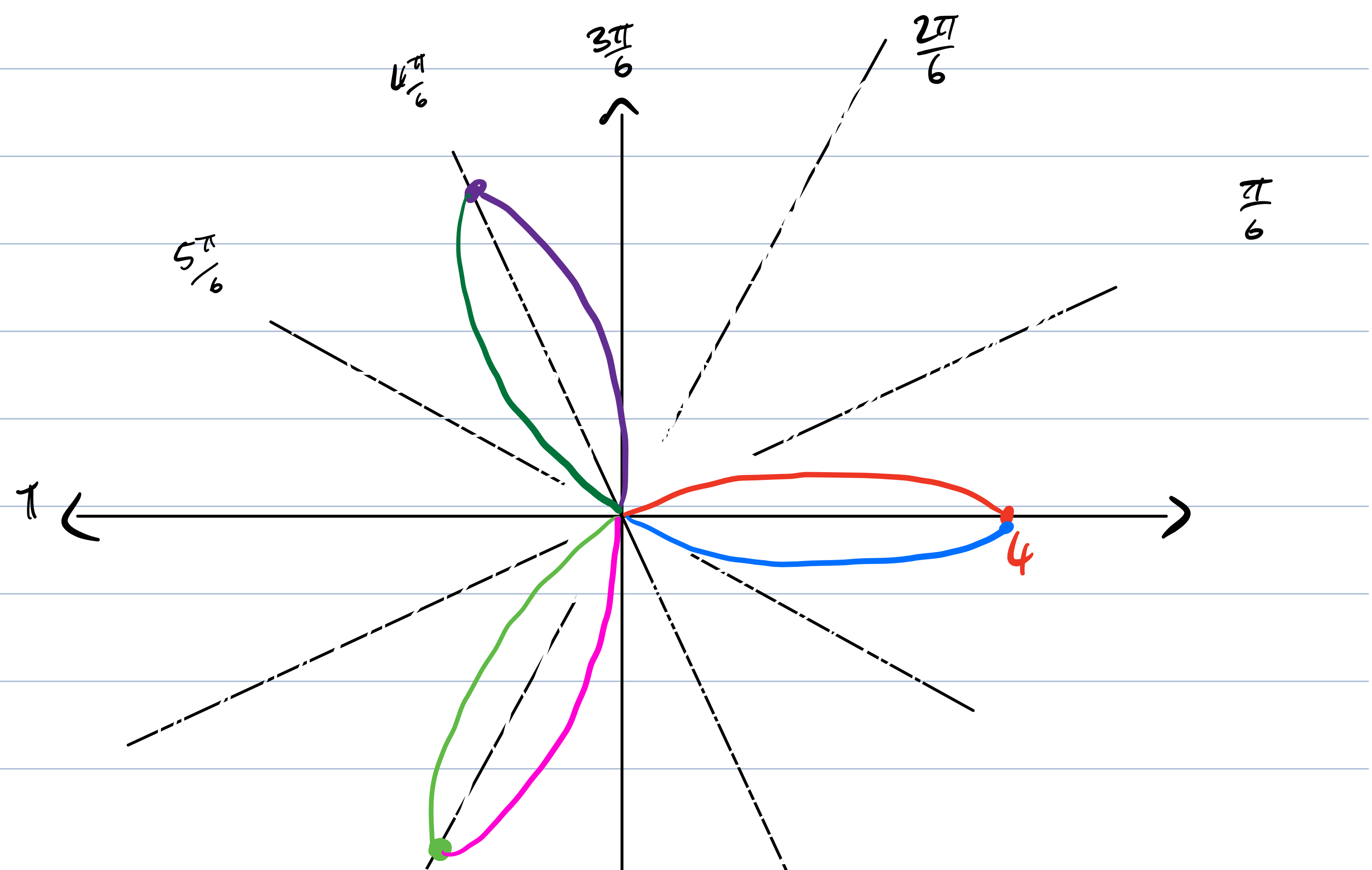
$\theta = \frac{2\pi}{3}$

Draw Rose Pedal Polar Curve

$r = 4\cos(3\theta)$

θr-plane





5. Which of the following integrals gives the area of the region enclosed by one petal of the polar curve $r = 4\cos(3\theta)$?

$4\cos(3\theta) = 0 \Rightarrow 3\theta = -\pi/2, \pi/2, \ldots \Rightarrow \theta = -\pi/6, \ldots$

$4\cos(3\theta) = 0 \Rightarrow \cos(3\theta) = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$

sector $\frac{1}{2}r^2\theta$

$\pi r^2 \left(\frac{\theta}{2\pi}\right)$

$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} f(\theta)^2\,d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 16\cos^2(3\theta)\,d\theta$$

$\cos^2(\alpha) = \frac{1}{2} + \frac{\cos(2\alpha)}{2}$

$$= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\,d\theta$$

$$= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} + \frac{\cos(6\theta)}{2}\,d\theta$$

E

$$= 4\left[\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1\,d\theta + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos(6\theta)\,d\theta\right]$$

$\sin(6\theta)$
$= \cos(6\theta) \cdot 6$

$$= 4 \cdot \frac{2\pi}{6} + \frac{1}{6}\sin(6\theta)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{4\pi}{3} + \frac{1}{6}\left(\sin(\pi) - \sin(-\pi)\right)$$

$$= \frac{4\pi}{3}$$

Question 6 : CI

$x(t) = 3t^2 - 5$

$y(t) = 4t^2 + 1$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$a = \sqrt{2}$, $b = \sqrt{5}$, $\frac{dx}{dt} = 6t$

$$\frac{dy}{dt} = 8t$$

$$L = \int_{\sqrt{2}}^{\sqrt{5}} \sqrt{(6t)^2 + (8t)^2}\, dt$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \sqrt{(36+64)t^2}\, dt$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \sqrt{100}\sqrt{t^2}\, dt$$

$$= 10 \int_{\sqrt{2}}^{\sqrt{5}} t\, dt$$

$$= 10 \left.\frac{t^2}{2}\right|_{\sqrt{2}}^{\sqrt{5}}$$

$$= 5(5-2)$$

$$= 5 \cdot 3$$

$$= 15$$

D

Question 7: (Integration technique (A.I.))

(Both odd)

Evalute $\int \sec^5 x \tan^3 dx$

$s^2 + c^2 = 1$

$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2}$

- $\frac{d}{dx} \tan(x) = \sec^2(x)$

$\tan^2 x + 1 = \sec^2 x$

- $\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$

$\tan^2(x) = \sec^2(x) - 1$

- $\int \sec(x)^n \left(\sec(x)\tan(x)\right)$

$= \frac{\sec^n(x)}{n+1} + C$

Idea.

- $\int \sec^5(x) \cdot \left(\sec^2(x) - 1\right) \tan(x)\, dx$

$= \int \sec^7(x)\tan(x)\, dx - \int \sec^5(x)\tan(x)\, dx$

$= \int \sec^6(x)\left[\sec(x)\tan(x)\right] dx - \int \sec^4(x)\left[\sec(x)\tan(x)\right] dx$

$= \frac{\sec^7(x)}{7} - \frac{\sec^5(x)}{5} + C$ A

- What about

$$\int \sec^5(x)\tan^2(x)\,dx \,?$$

$\tan^2 x + 1 = \sec^2$

$$= \int \sec^5(x)\left(\sec^2(x) - 1\right) dx$$

$$= \int \sec^7(x)\,dx - \int \sec^5(x)\,dx.$$

- What about

$$\int \sec^4(x)\tan^2(x)\,dx \,?$$

(Both even)

$$= \int \left(\tan^2(x)+1\right)\tan^2(x)\sec^2(x)\,dx$$

$$= \int \tan^4(x)\cdot\sec^2(x)\,dx + \int \tan^2(x)\sec^2(x)\,dx$$

$$= \frac{\tan^5(x)}{5} + \frac{\tan^3(x)}{3} + C.$$

Question 8 :

BI

- Evaluate
- Determine conv & Find sum.

$$\sum_{n=0}^{\infty} \frac{2}{n^2+3n+2}$$

• Finding sum
  - Geometric series
  - Telescopic series.
    - PFD

Telescopic?

(*) See Q13 too

$$n^2+3n+2 = (n+1)(n+2)$$

So $$\frac{2}{n^2+3n+2} = \frac{2}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}.$$

Then $$\frac{2}{(n+1)(n+2)} = \frac{A(n+2) + B(n+1)}{(n+1)(n+2)}$$

$$= \frac{(A+B)n + 2A + B}{(n+1)(n+2)}$$

$\Rightarrow$ $A = -B$, $2A + B = 2$ $\Rightarrow$ $2A - A = 2$

$\Rightarrow$ $\boxed{A = 2}$

$\Rightarrow$ $\boxed{B = -2}$

So $$\sum_{n=0}^{\infty} \frac{2}{n^2+3n+2} = \sum_{n=0}^{\infty} \left(\frac{2}{n+1} - \frac{2}{n+2}\right)$$

$$= \sum_{n=0}^{\infty} (b_n - b_{n+1})$$ where

key!

$(b_n) = \left(\frac{2}{n+1}\right)$. By telesopic series

$$\sum_{n=0}^{\infty}(b_n-b_{n+1}) = b_0 - \lim_{n\to\infty} b_{n+1}$$

$$= 2 - \lim_{n\to\infty}\frac{2}{n+2}$$

$$= 2$$

So $\boxed{\sum_{n=0}^{\infty}\frac{2}{n^2+3n+2} = 2}$ C

Recall Telescopic Series:

Let (b_n) be a sequence, and consider the series of the form

$$\sum_{n=1}^{\infty}(b_n-b_{n+1}).$$ Then

$$\sum_{n=1}^{\infty}(b_n-b_{n+1}) = b_1 - \underline{\lim_{n\to\infty} b_{n+1}}$$

exists or not.

- So series converge iff sequence (b_n) conv.

Question 9 : (Power series Rep)

Find $$\int x \ln(1+x^3)\,dx.$$

- Build new from old:

- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, $(-1,1]$

- $\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}$, $(-1,1]$

- $x\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n+1}}{n}$, $(-1,1]$

- $\int x\ln(1+x^3)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \frac{x^{3n+2}}{(3n+2)}$, at least $(-1,1]$ could be $[-1,1]$

term wise

So / B

Question 10 :

C II

10. Find the slope of the line tangent to the polar curve $r = \cos(2\theta)$ when $\theta = \frac{\pi}{2}$.

Treat as a parameterization

$x = r\cos(\theta) = f(t)\cos(t)$

$y = r\sin(\theta) = f(t)\sin(t)$

$f(t) = \cos(2t)$

$f'(t) = -2\sin(2t)$

So $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

So $\boxed{\frac{dy}{dx} = \frac{0}{-1} = 0}$

$\frac{dy}{dt} = f'(t)\sin(t) - f(t)\cos(t)$

$= -2\sin(2t)\cdot\sin(t) - \cos(2t)\cdot\cos(t)$

$\left.\frac{dy}{dt}\right|_{\frac{\pi}{2}} = -2\sin\left(2\cdot\frac{\pi}{2}\right)\cdot\sin\left(\frac{\pi}{2}\right) - \cancel{\cos(\ )\cdot\cos\left(\frac{\pi}{2}\right)}$

$= 0$

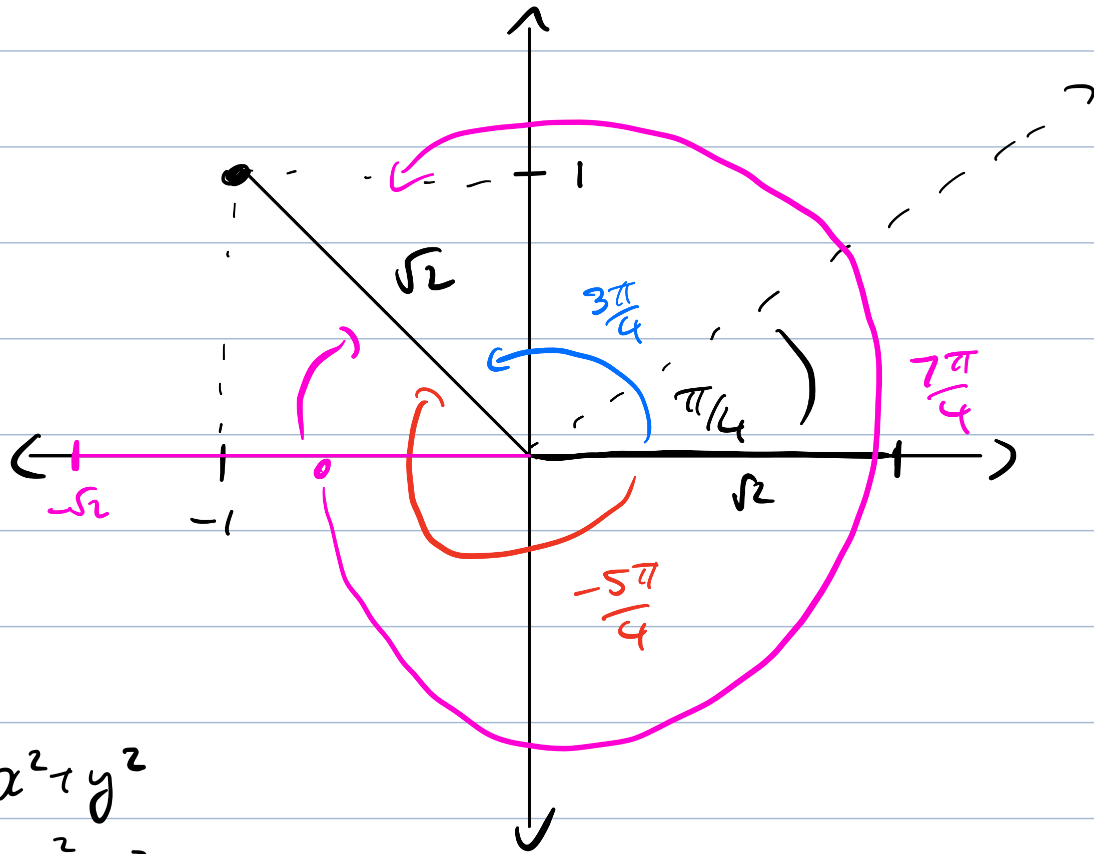
$\frac{dx}{dt} = f'(t)\cos(t) + f(t)\sin(t)$

$= \cancel{-2\sin(2t)\cdot\cos(t)}^{0} + \cos(2t)\sin(t)$

$\frac{dx}{dt} = -1$

Question 11 : CI

Given Cartesian / Rectangular co-ordinates $(-1, 1)$, find its Polar co-ordinates.



$r^2 = x^2 + y^2$

$= (-1)^2 + (1)^2 = 2$

$\Rightarrow r = \pm\sqrt{2}$.

- $y = r\sin(\theta) \Rightarrow 1 = \pm\sqrt{2}\sin\theta \Rightarrow \sin(\theta) = \pm\frac{1}{\sqrt{2}}$

Ref angle : $\frac{\pi}{4}$

S

- $(\sqrt{2}, \frac{3\pi}{4})$, $(\sqrt{2}, \frac{-5\pi}{4})$,

- $(-\sqrt{2}, \frac{-\pi}{4})$, $(-\sqrt{2}, \frac{7\pi}{4})$,

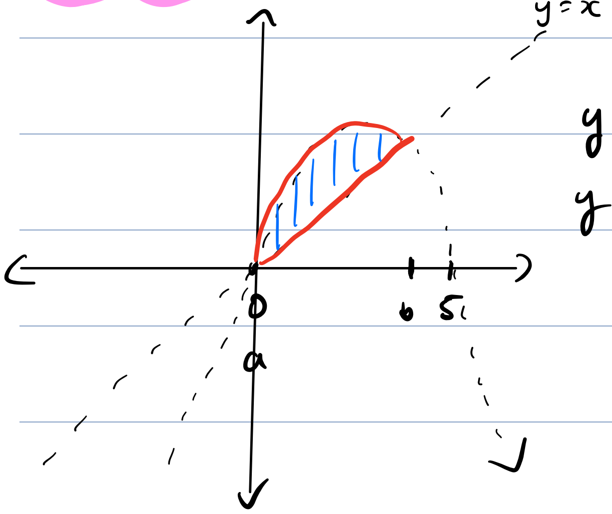
R

R, S only

E

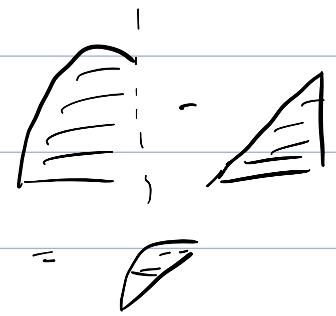
Question 12 :

AI — Area of region bounded by curve.



$y = 5x - x^2 = x(5-x)$

$y = x$



① Find bounds of integration

- $5x - x^2 = x \Rightarrow 0 = x^2 + x - 5x$

$\Rightarrow 0 = x^2 - 4x = x(x-4)$

- Bounds are $[0,4]$.

$$\int_0^4 5x - x^2\,dx - \int_0^4 x\,dx$$

$$= \frac{5x^2}{2} - x^3/3 \Big|_0^4 - \frac{x^2}{2}\Big|_0^4$$

$$= \frac{5}{2}(4)^2 - \frac{4^3}{3} - \frac{4^2}{2}$$

$$= 4^2\left(\frac{5}{2} - \frac{4}{3} - \frac{1}{2}\right) = 16\left(\frac{15}{6} - \frac{8}{6} - 3/6\right)$$

$$= 16\left(\frac{4}{6}\right)$$

$$= 16 \cdot \frac{2}{3}$$

$$= \frac{32}{3}$$

Question 13:

B I

Evaluate $\sum_{n=2}^{\infty} \frac{(-2)^{n+1}}{3^{n-2}}$

- What is difference betw this and $\sum \frac{2}{n^2+3n+2}$?

See Q8

- Here we are Geometric.

$$= \sum_{n=2}^{\infty} \frac{(-2) \cdot (-2)^n}{3^{-2} \cdot 3^n}$$

$$= (-2) \cdot 3^2 \sum_{n=2}^{\infty} \left(\frac{-2}{3}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{1}{1+2/3} = \frac{1}{5/3} = 3/5$$

$$= (-18)\left(3/5 - 1 - \left(-\frac{2}{3}\right)\right)$$

$$= (-18)\left(9/15 - \frac{15}{15} + \frac{10}{15}\right)$$

$$= (-18)\left(\frac{19}{15} - \frac{15}{15}\right)$$

$$= \frac{(-18) \cdot (4)}{15}$$

$$= \frac{\cancel{3} \cdot (-6 \cdot 4)}{\cancel{3} \cdot 5} = \frac{-24}{5}$$

E

Question 14: $\boxed{B\,\text{II}}$ Taylor & Mclaurin

Given $f(0)=2$, $f'(0)=-1$, $f''(0)=3$, $f'''(0)=-4$. Find Taylor series about 0.

Given a function f that has a Taylor series about a, we know

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}.$$

So find $\displaystyle\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^{n}$

$$=\frac{2}{0!}x^{0}+\frac{-1}{1!}x^{1}+\frac{3}{2!}x^{2}+\frac{-4}{3!}x^{3}$$

$$=2-x+\frac{3}{2}x^{2}-\frac{2}{3}x^{3}.$$

B

Question 15

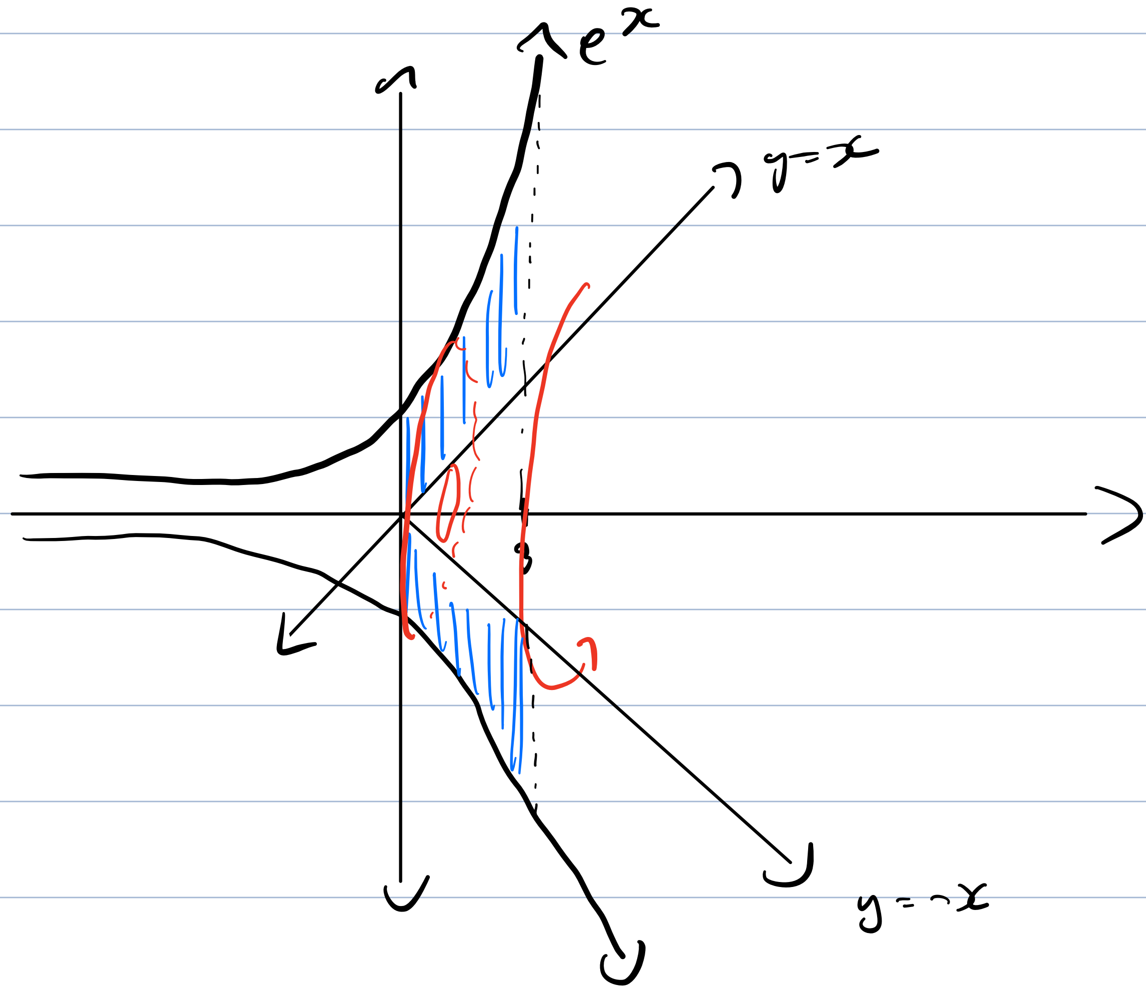
⊛ Volume of revolution.

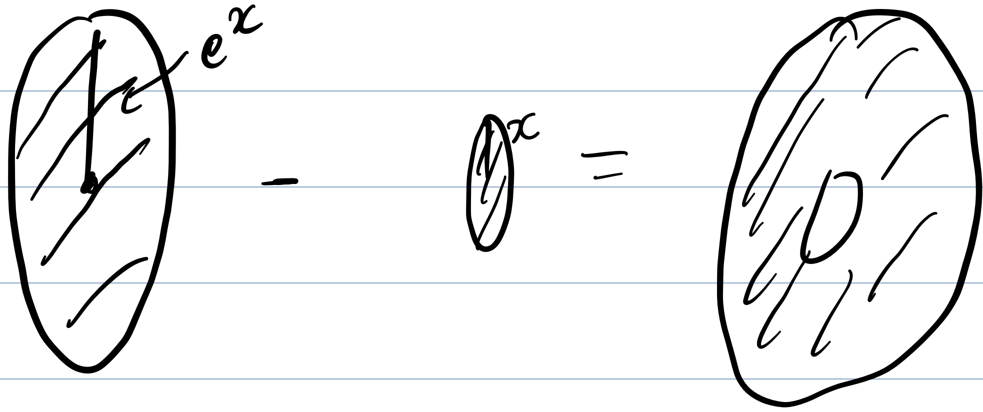
R region bounded by

- $y = e^x$
- $y = x$
- $x = 0$
- $x = 3$

washer method:

- Find volume of solid formed by rotating R about x-axis:





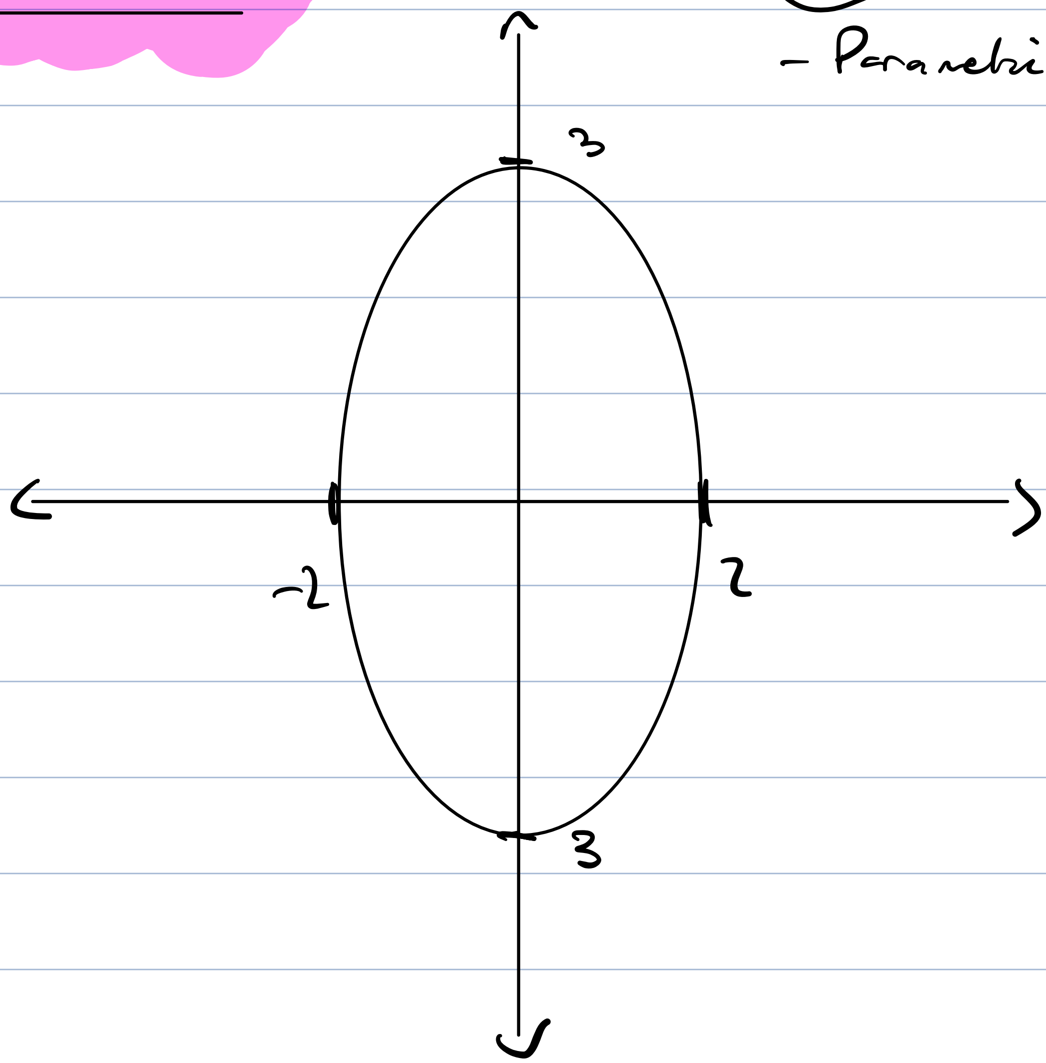
$$\int_0^3 \pi (e^x)^2 \, dx - \int_0^3 \pi x^2 \, dx$$

$$= \pi \int_0^3 e^{2x} - x^2 \, dx$$ A

Question 16:

CI

- Parametric equations.

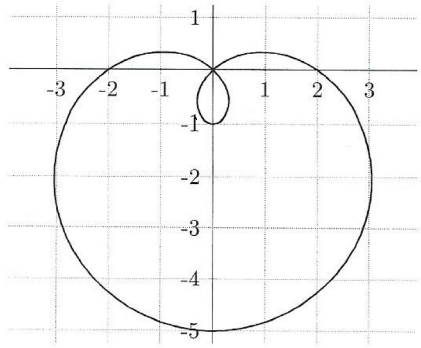


$$(x(t), y(t)) = (2\cos(t), 3\sin(t))$$ C

Think of unit circle:

- $x = \cos(\theta)$, $y = \sin(\theta)$, $\theta \in [0, 2\pi)$

Question 17:



Options:

- $r = 2 - 3\sin\theta$ (A)
- $r = 3 + 2\sin\theta$ (B) ✗
- $r = 3 - 2\sin\theta$ (C)
- $r = 2 + 3\sin\theta$ (D) ✗

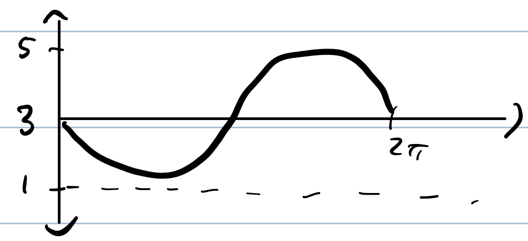
① Maximum?

- (B) and (D) reaches maxima at $(0, 5)$

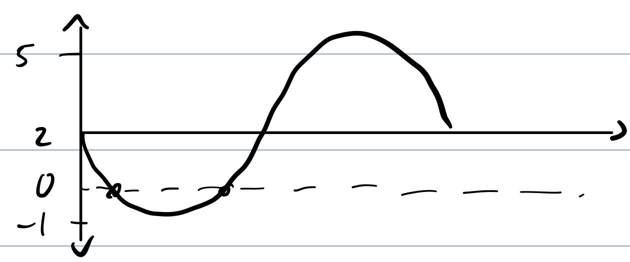
② Difference between? # zeroes?

- $r = 2 - 3\sin\theta$ (hits zero rad)
- $r = 3 - 2\sin\theta$? (never hit 0 radius)

$r = 3 - 2\sin\theta$



$r = 2 - 3\sin\theta$



So must be $r = 3 - 2\sin\theta$. (A)

Question 18 :

A.I

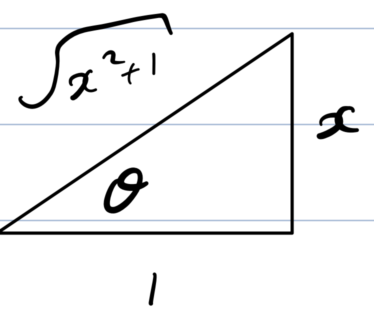
Improper Integrals

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}\,dx$$

- u-sub?
- IBP ?
- PFD ?
- Trig sub? (Yes)

- $\sin^2(x) = 1 - \cos^2(x)$
- $\sec^2(x) = \tan^2(x) + 1$
- $\tan^2(x) = \sec^2(x) - 1$



- $x = \tan(\theta) \Rightarrow dx = \sec^2(\theta)\,d\theta$

$$\int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2(\theta)}{1+\tan^2(\theta)}\,d\theta = \int \frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta$$

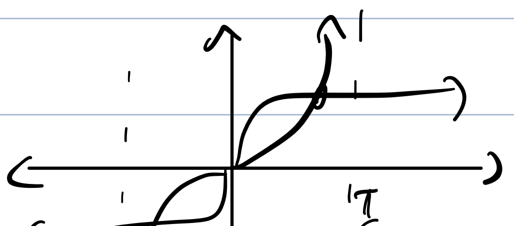
$$= \int d\theta$$

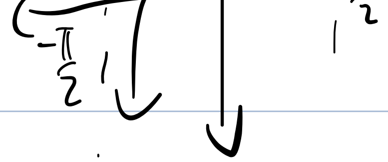
$$= \theta.$$

$$= \arctan(x)\Big|_1^t$$

$$= \arctan(t) - \arctan(1)$$

- $x = \tan(\theta) \Rightarrow \theta = \arctan(x).$





$= \arctan(t) - \frac{\pi}{4}$

$\tan(\frac{\pi}{4}) = 1$ iff

$\arctan(1) = \frac{\pi}{4}$.

So $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(t) - \frac{\pi}{4}$

$= \frac{2\pi}{4} - \frac{\pi}{4} = \frac{\pi}{4}$

D

Question 19:

B.I. Convergence, divergence of series.

A: $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n+1}$ — conv – p series, DCT.

B: $\sum_{n=2}^{\infty} \frac{\sin^2(n)}{n^2+n}$ — conv – p series, DCT, $|\sin^2(n)| \leq 1$

C: $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$

•) A: $0 \leq \frac{1}{n^{3/2}+n+1} \leq \frac{1}{n^{3/2}}$.

– So A conv, by DCT and p series, $p>1$.

•) B $0 \leq \frac{\sin^2(n)}{n^2+h} \leq \frac{1}{n^2}$

•) C: $\frac{1}{n} \leq \frac{\ln(n)}{n}$, so div by DCT & Harmonic series.

or Generalized p-series

$$\sum \frac{1}{n^p \ln(n)^q} = \begin{cases} p>1, & \text{conv} \\ p=1, & q>1 \text{ conv} \\ p<1, & \text{div} \\ p=1, & q \leq 1 \text{ div} \end{cases}$$

$\rightarrow \frac{\ln(n)}{n} = \frac{1}{n \cdot \ln(n)^{-1}} = \begin{cases} p=1, q=-1 \\ \text{div.} \end{cases}$

Question 20:

AST Estimati:

- Given $f(x) = \cos(x)$
- Find max error in using Taylor series about 0 of f to estimate $\cos(\frac{1}{2})$.

Use AST estimation.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Use first two n-zero - terms:

$$\text{So } \cos(\tfrac{1}{2}) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(\frac{1}{2})^{2n}}{2n!} = \sum_{n=0}^{\infty} (-1)^n a_n.$$

$$\text{Error} = \left| \cos(\tfrac{1}{2}) - \sum_{n=0}^{1} (-1)^n \frac{(\frac{1}{2})^{2n}}{(2n)!} \right|$$

$$\leq \frac{(-1)^2 (\frac{1}{2})^{2(2)}}{(2(2))!}$$

$$= \frac{1}{2^4 \cdot 4!}$$

$$= \frac{1}{2^7 \cdot 3}$$

$$= \frac{1}{384} \quad E$$

$4! = 4 \cdot 3 \cdot 2$
$= 2^3 \cdot 3$

2^7

$$\begin{array}{r} 128 \\ \times\ 3 \\ \hline 384 \end{array}$$

1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256

Question 21 :

CI

Find slope of Tagent line to a parametric curve

$x(t) = 3t^2 + \sqrt{t}$

$y(t) = 5t^3 - t - \ln(t)$

when $t = 1$.

① $\frac{dx}{dt} = 6t + \frac{1}{2}t^{-\frac{1}{2}}$, $\frac{dy}{dt} = 15t^2 - t - \frac{1}{t}$

② Slope at $t=1$ is

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{\left.\left(\frac{dy}{dt}\right)\right|_{t=1}}{\left.\left(\frac{dx}{dt}\right)\right|_{t=1}} = \frac{15-2}{6+\frac{1}{2}} = \frac{13}{13/2}$$

$$= 13 \cdot \frac{2}{13}$$

$$= 2.$$

Question 22 :

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$$

What does the ratio test say?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{\cancel{2} \cdot \cancel{4} \cdot \cancel{6} \cdots \cancel{(2n)}\,(2(n+1))}{\cancel{1} \cdot \cancel{4} \cdot \cancel{7} \cdots \cancel{(3n-2)}\,(3(n+1)-2)} \cdot \frac{\cancel{1} \cdot \cancel{4} \cdot \cancel{7} \cdots \cancel{(3n-2)}}{\cancel{2} \cdot \cancel{4} \cdot \cancel{6} \cdots \cancel{(2n)}}$$

$3n+3-2 = 3n+1$

$$= \lim_{n\to\infty} \frac{2n+2}{3n+1}$$

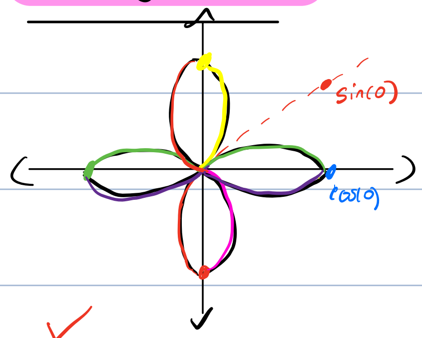
$$= \lim_{n\to\infty} \frac{2+\frac{2}{n}}{3+\frac{1}{n}} = \frac{2}{3} < 1$$

Hence by ratio-test, we have conv, B

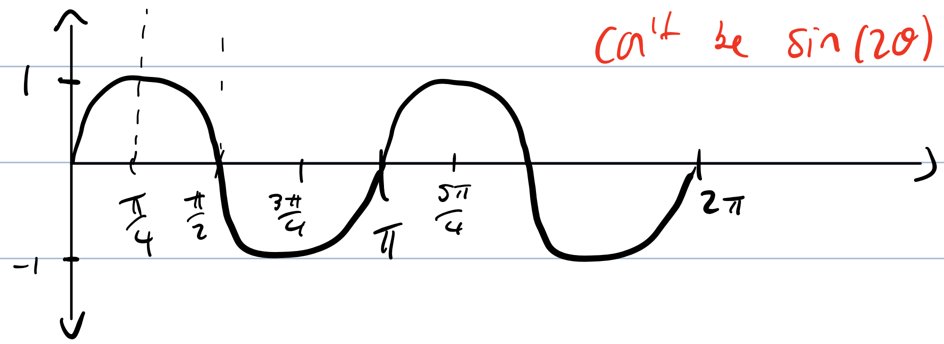
Final Exam
Fall 2023

Question 4

(C.1)

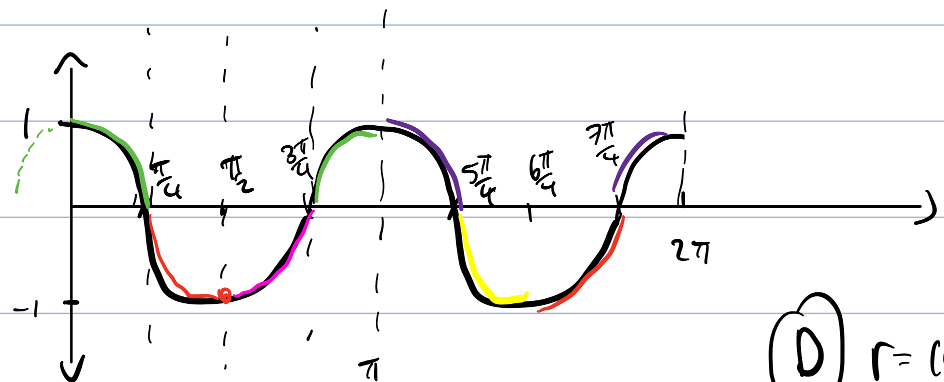


✗ $r = \sin(2\theta)$, $2\theta = 2\pi \Rightarrow \theta = \pi$



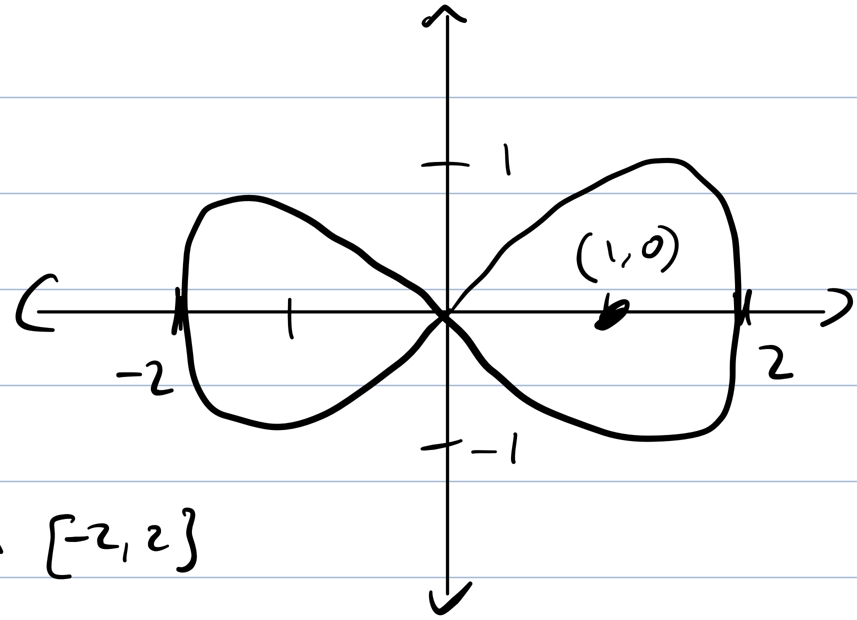
can't be sin(2θ), cos(2θ)

✓ $r = \cos(2\theta)$



(D) $r = \cos(2\theta)$

Question 11 :



(i) x's vary between $[-2,2]$

So cannot be

- $x = \cos^3(\theta)$ (A not possible)

(2) y's vary between $[-1,1]$ so

- $y = 4\sin(\theta)\cdot\cos(\theta) = 2\sin(2\theta)$ (D not possible)

For B,

$x(0) = \cos(0) + 0 \cdot \sin(0) = 1$

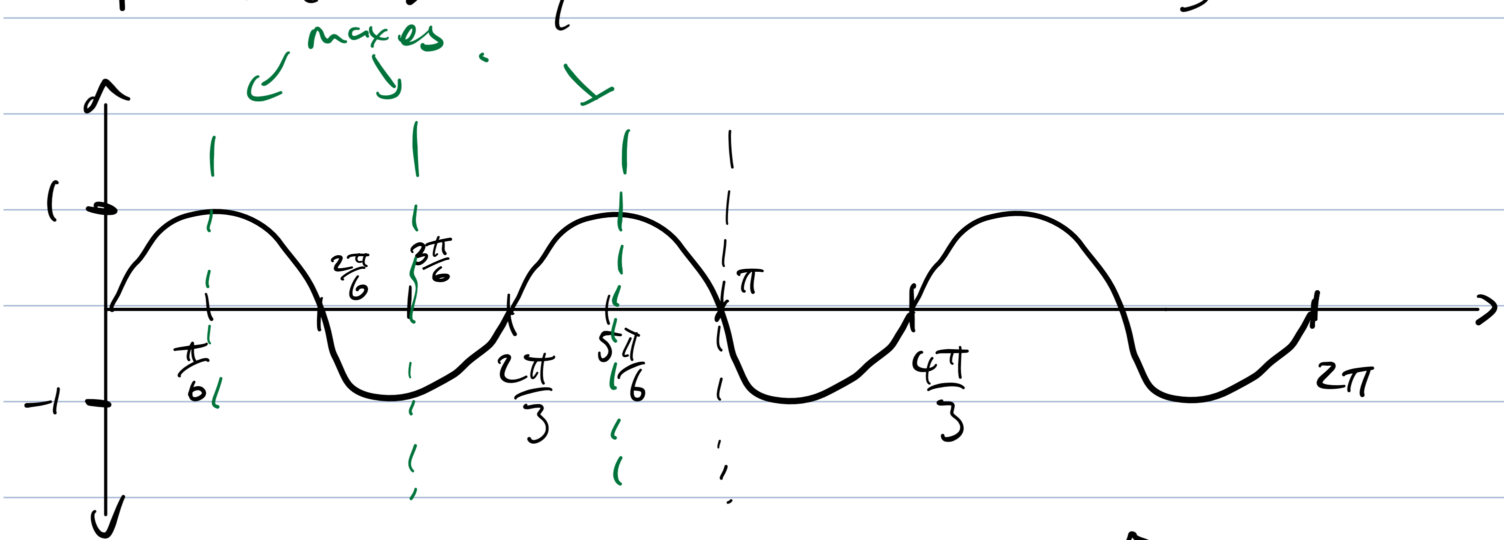
$y(0) = \sin(0) - 0 \cdot \cos(0) = 0$

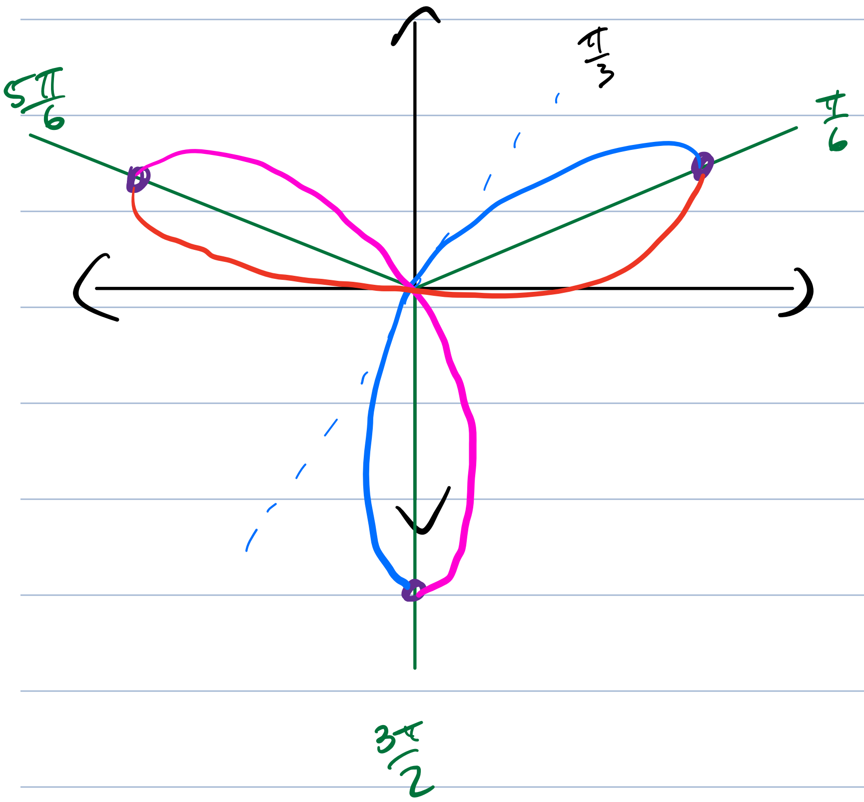
But $(1,0)$ not on curve, so B not possible.

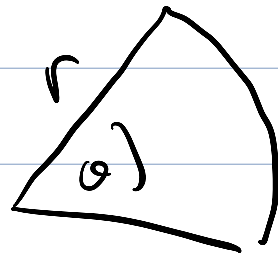
Hence C

Question 12 :

$r = \sin(3\theta)$ $\qquad 3\theta = 2\pi \Rightarrow \theta = \frac{2\pi}{3}$







$$\pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$$

$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} f(\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin^2(3\theta) \, d\theta$$

Sol D

Question 17 : (washer method)

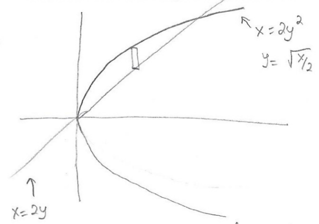
17. Let R be the region bounded by the curves $x = 2y^2$ and $x = 2y$. Which of the following integrals gives the volume of the solid formed when R is rotated about the x-axis using the Washer Method?

(A) $\pi \int_0^2 \left[\frac{x}{2} - \frac{x^2}{4}\right] dx$

(B) $\pi \int_0^2 \left[\sqrt{\frac{x}{2}} - \frac{x}{2}\right] dx$

(C) $\pi \int_0^1 \left[4y^2 - 4y^4\right] dy$

(D) $\pi \int_0^1 \left[2y - 2y^2\right] dy$



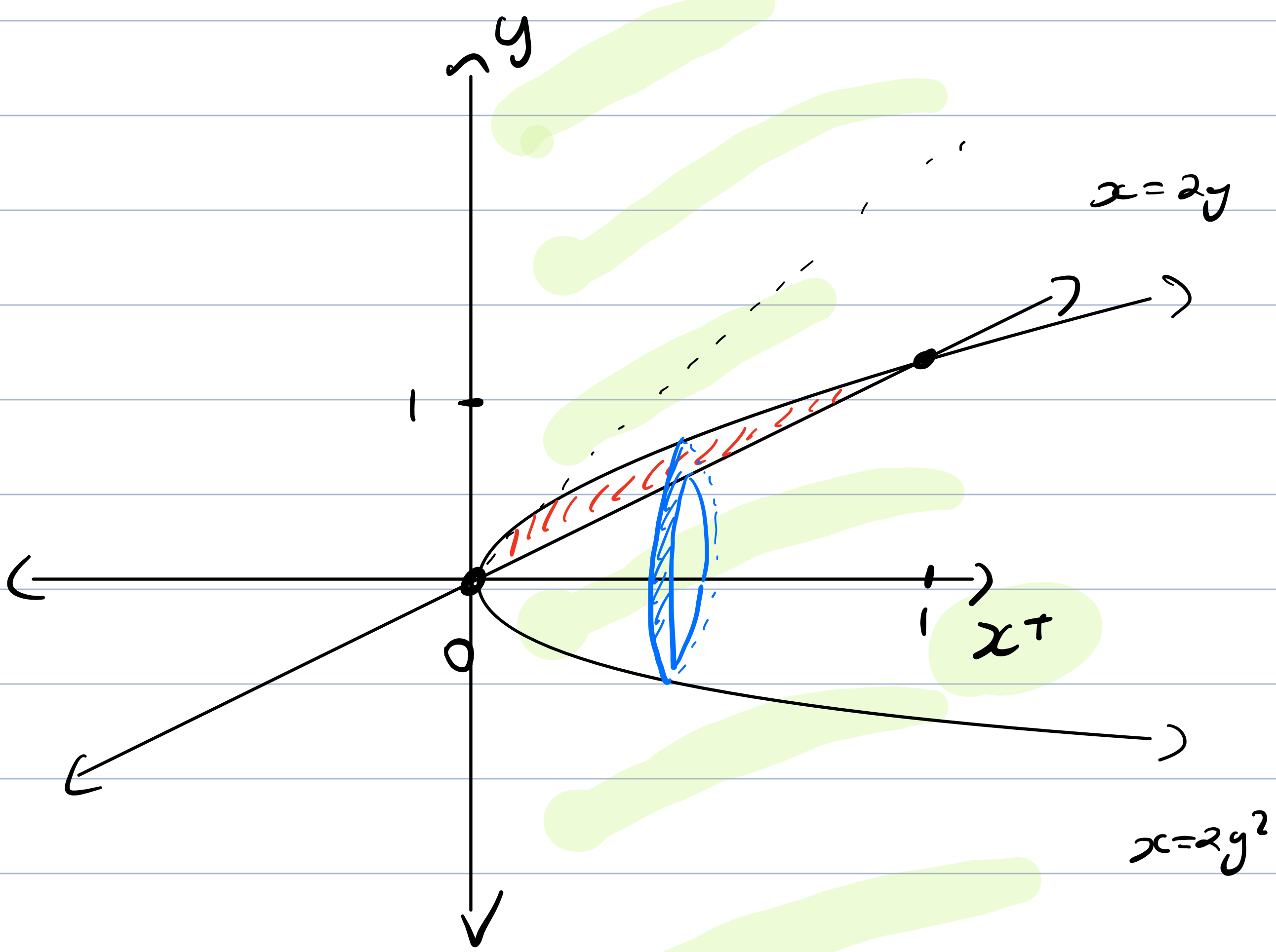
$2y = 2y^2 \Rightarrow 2y(y-1) = 0$
$\Rightarrow y = 0, 1$
$\Rightarrow x = 0, 2$

$V = \pi \int_0^2 \left[\left(\sqrt{x/2}\right)^2 - (x/2)^2 \right] dx = \pi \int_0^2 \left[x/2 - \frac{x^2}{4} \right] dx$

- $x = 2y^2$
- $x = 2y$

} R

- Find volume of solid formed by revolving R about x-axis.



- Washer method
- axis of rotation & integration axis be same!
- So $x = 2y^2 \Rightarrow \frac{x}{2} = y \Rightarrow y = \sqrt{\frac{x}{2}}$, $x \geq 0$, $y \geq 0$
- $x = 2y \Rightarrow y = \frac{x}{2}$.

So $\quad V = \int_0^1 \pi r_{out}^2 \, dx \quad - \quad \int_0^1 \pi r_{in}^2 \, dx$

$$= \int_0^1 \pi \sqrt{\frac{x}{2}}^2 \, dx \quad - \quad \int_0^1 \pi \left(\frac{x}{2}\right)^2 dx$$

$$= \pi \int_0^1 \left(\frac{x}{2} - \frac{x^2}{4}\right) dx$$

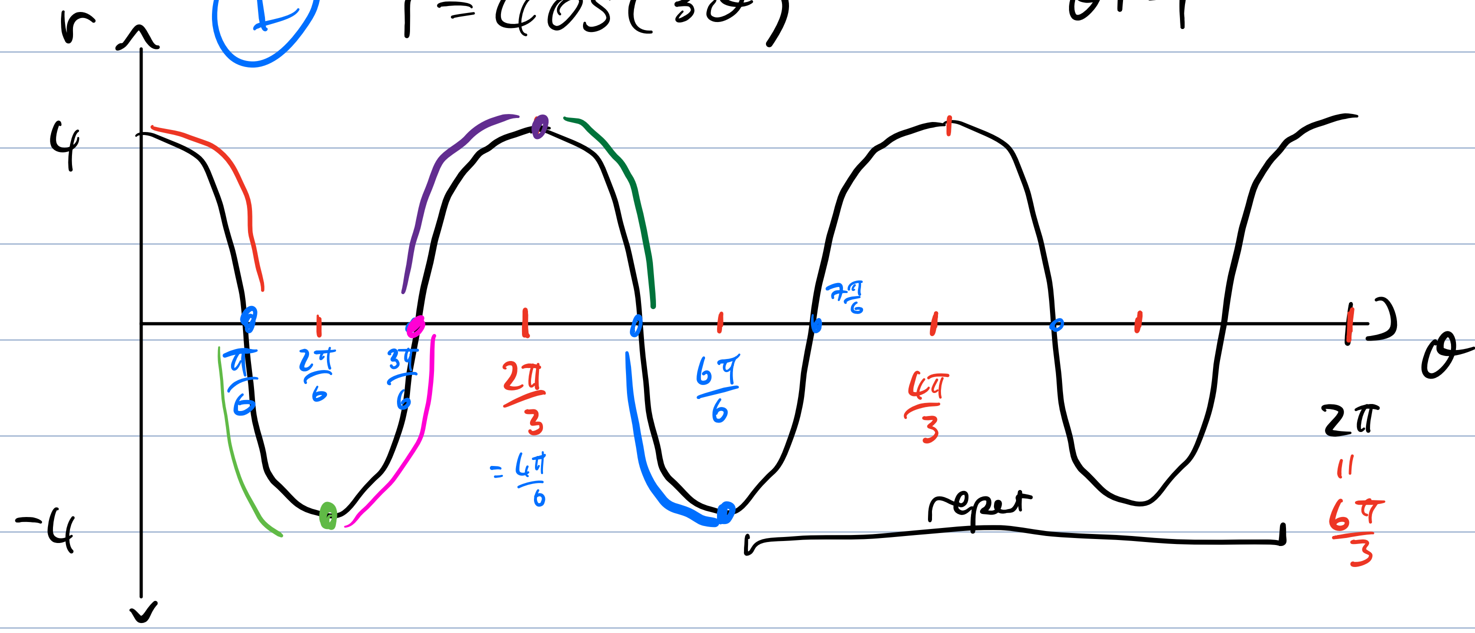
Selected

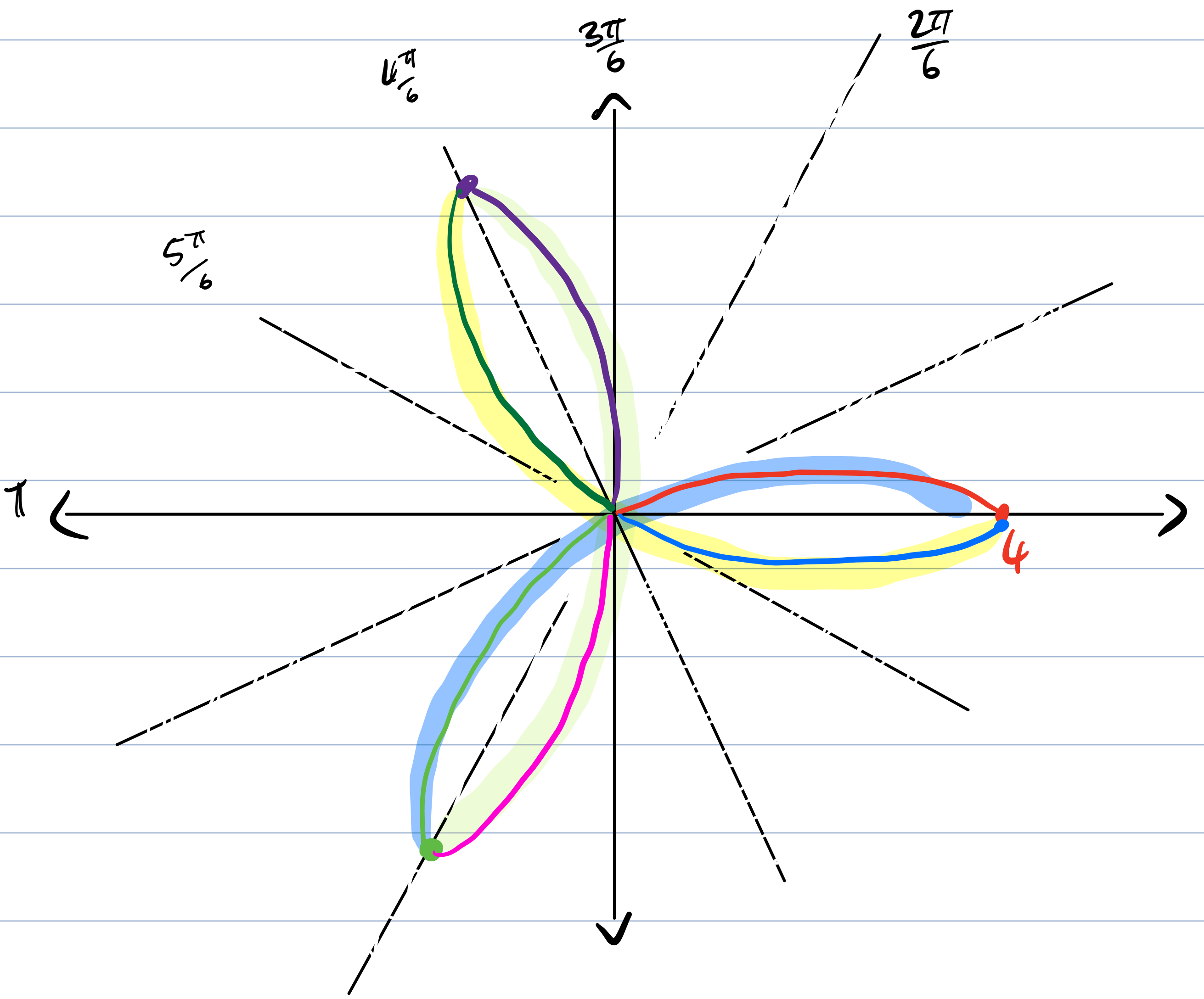
Topics

- Work
- Volume of Revolution.
  - → Disk
  - → Washer
  - → Shell
- Polar Area
- Derivation of arc length.

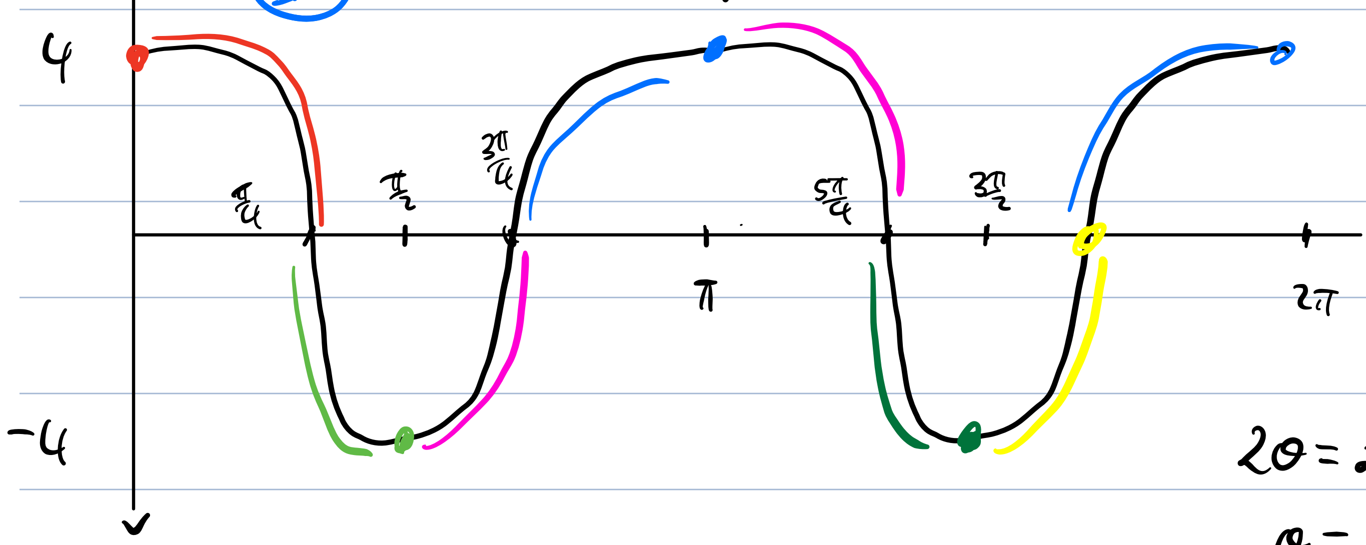
Example: $r = 4\cos(3\theta)$ vs $r = 4\cos(2\theta)$

I $\quad r = 4\cos(3\theta) \qquad \theta r$-plane



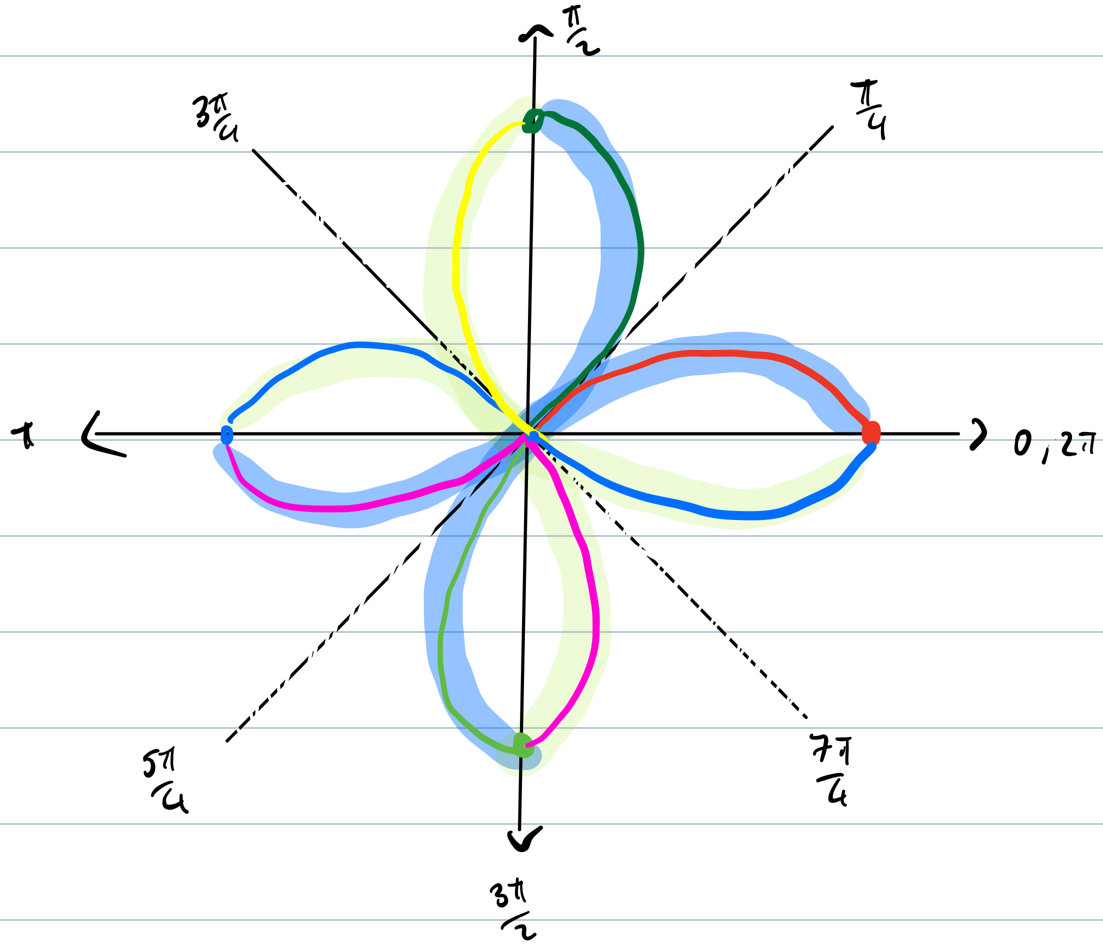


II $r = 4\cos(2\theta)$ θr-plane



$2\theta = 2\pi$

$\theta = \pi$



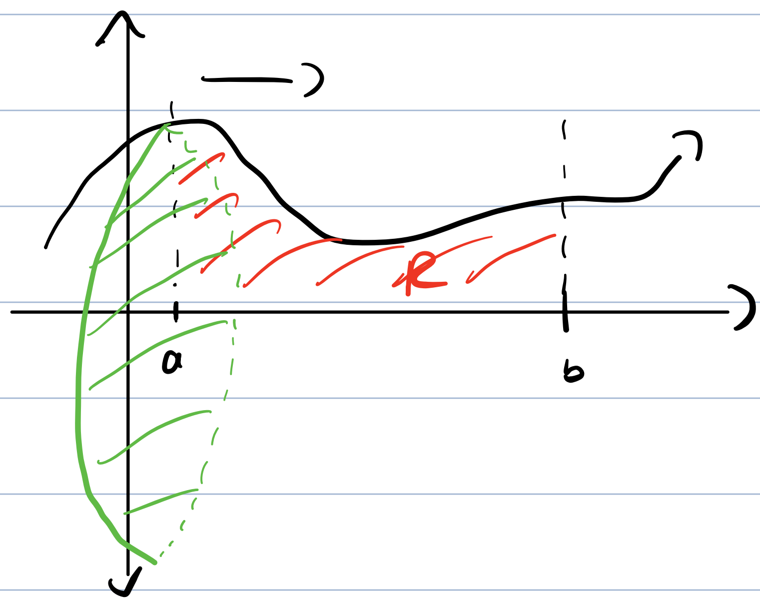
Areas & Volume

Types of Question:

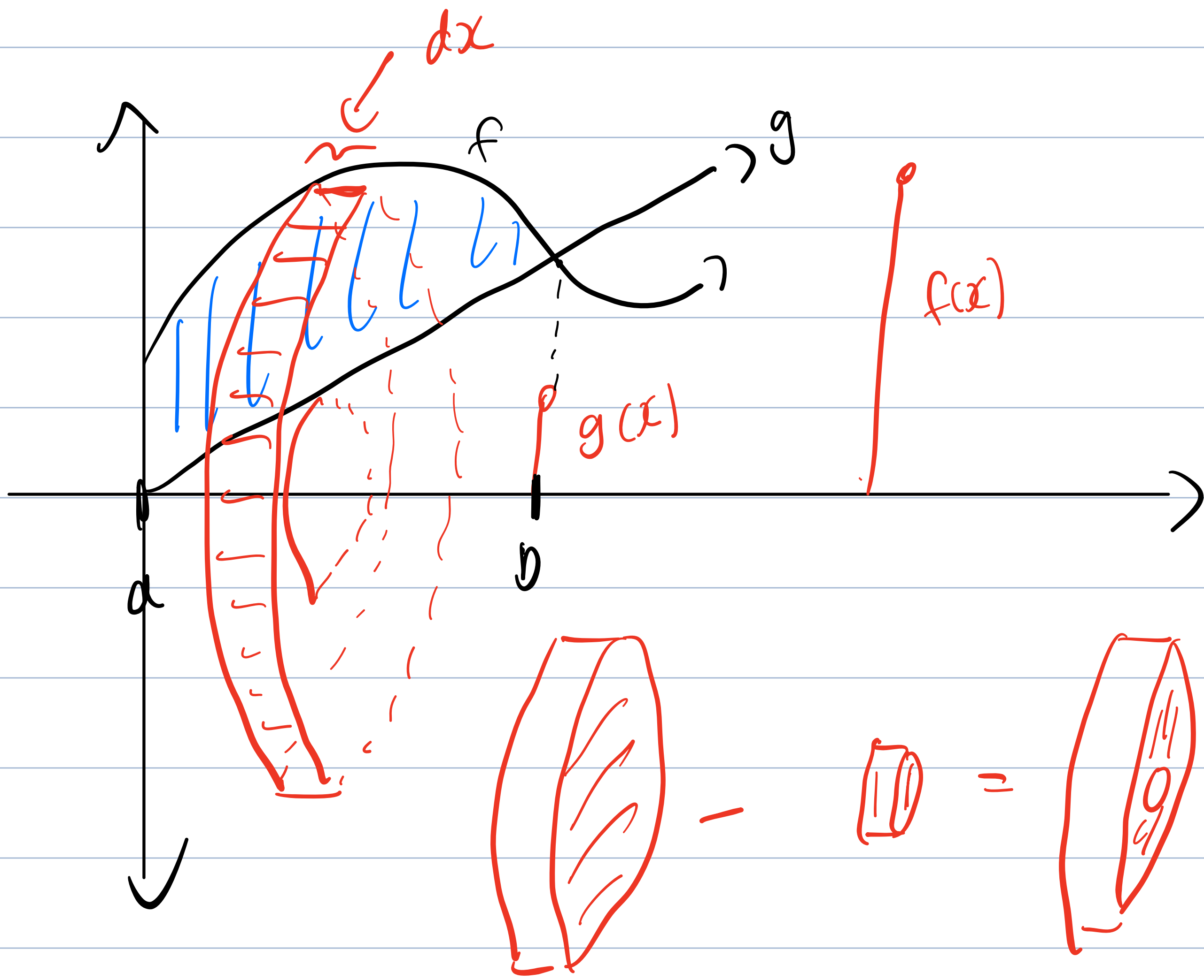
① Given curves that bound a region, find the area enclosed.

② Given a region, rotate it about x or y axis and find volume of revolution.

- Disk method
- Washer method
- Shell method



Recall washer method.

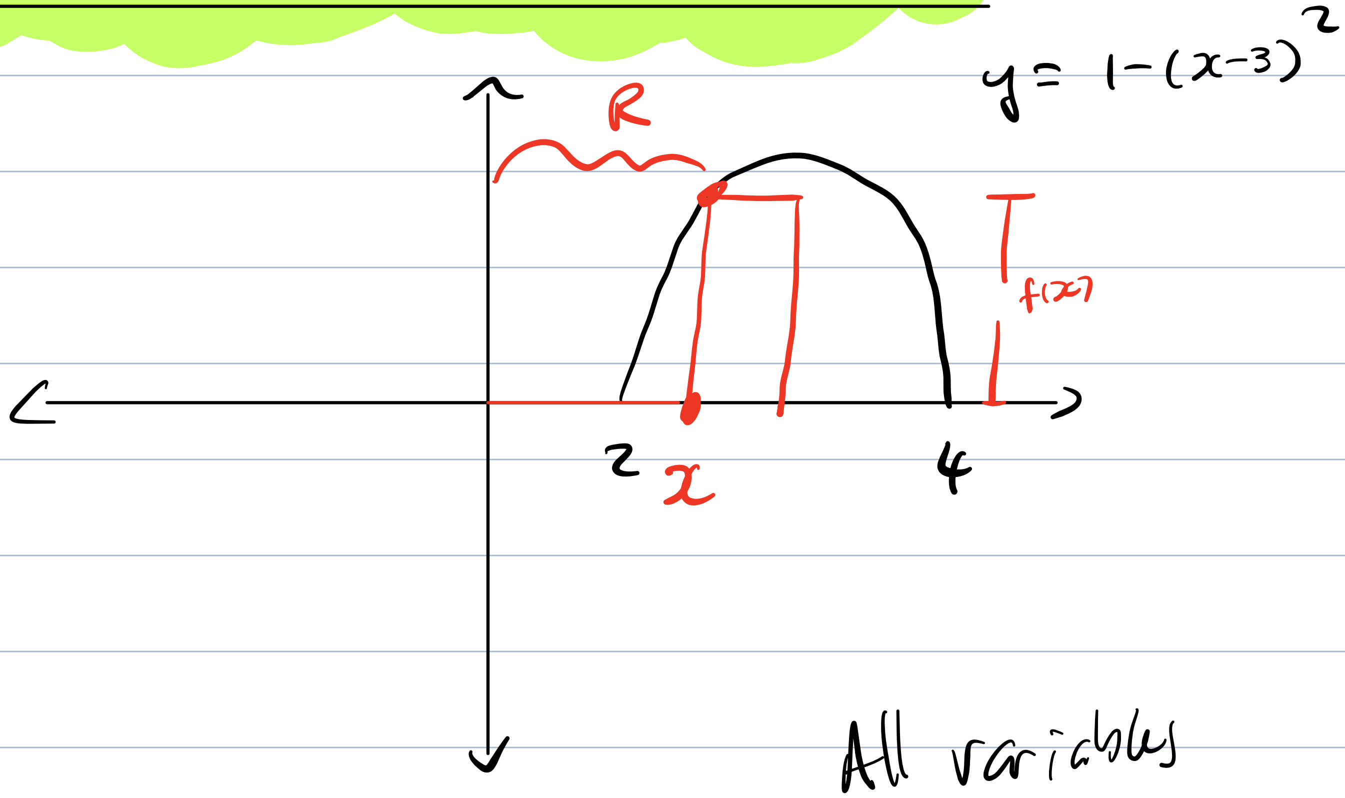


Cyllinder volume :

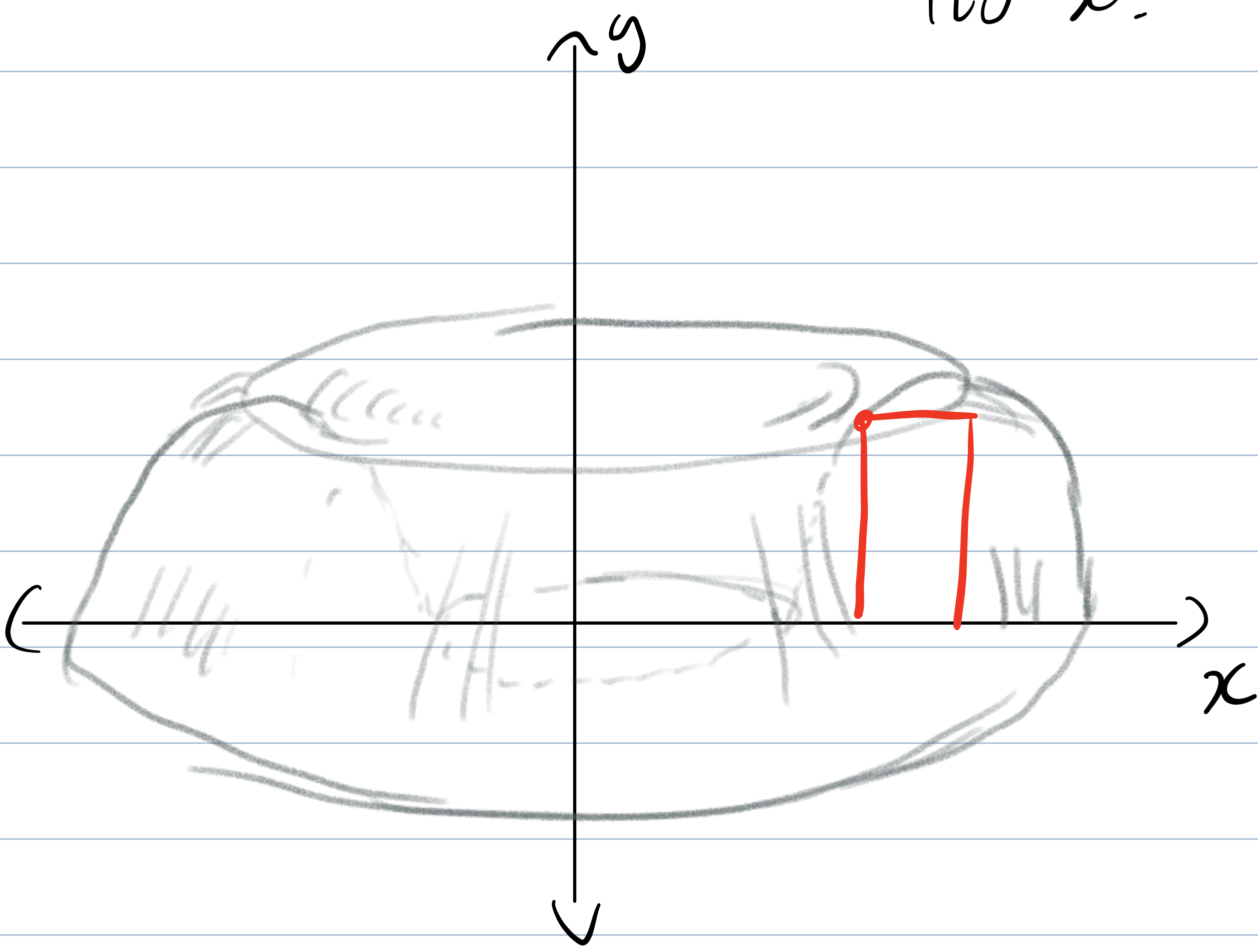
$$= \pi r_{outer}^2 \Delta x - \pi r_{inner}^2 \Delta x$$

$$= \pi f(x)^2 \Delta x - \pi g(x)^2 \Delta x$$

Explaining The Shell method:

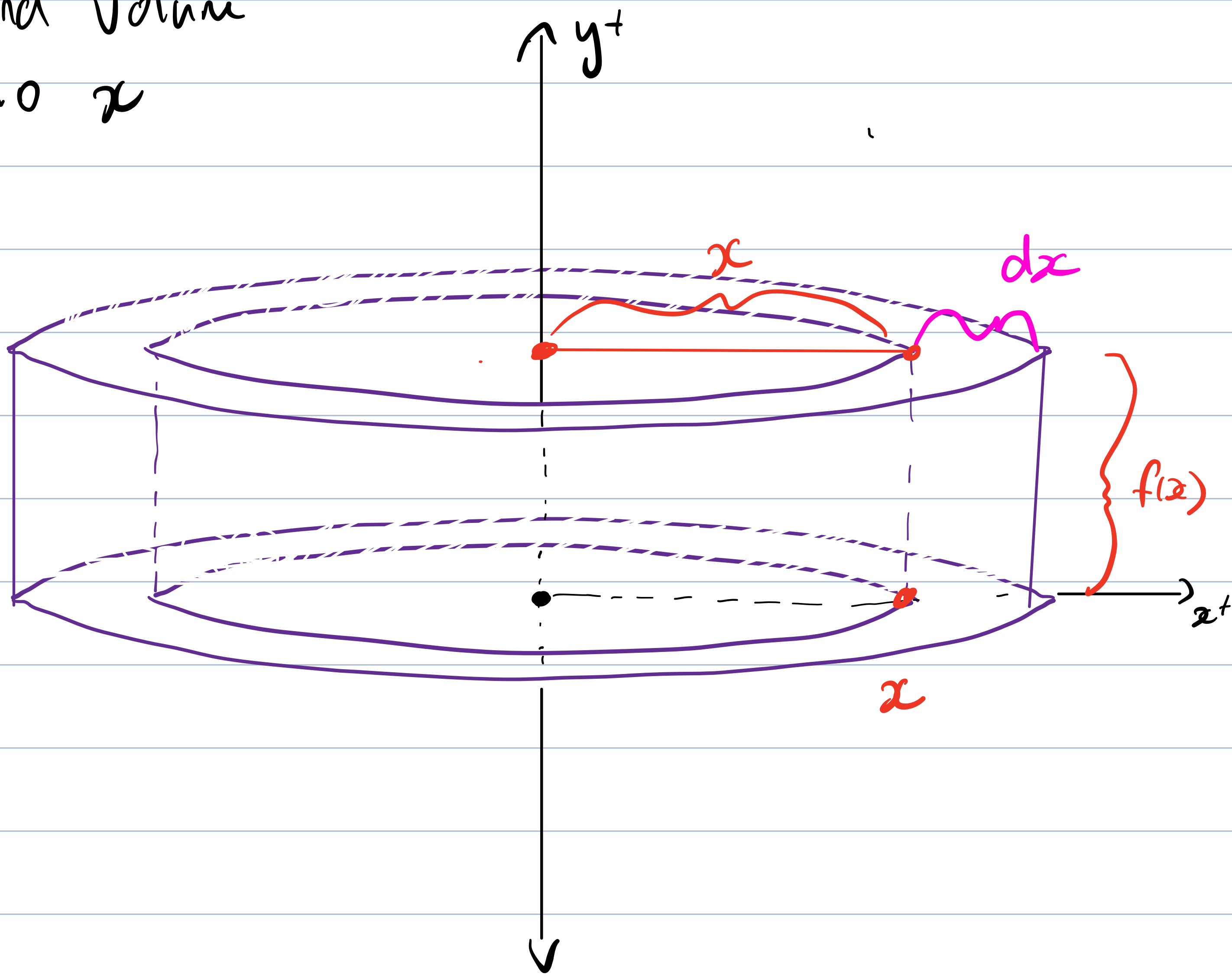


All variables ito x!



Find volume

i.t.o x



How to find volume of cylinder?

① outside surface area :

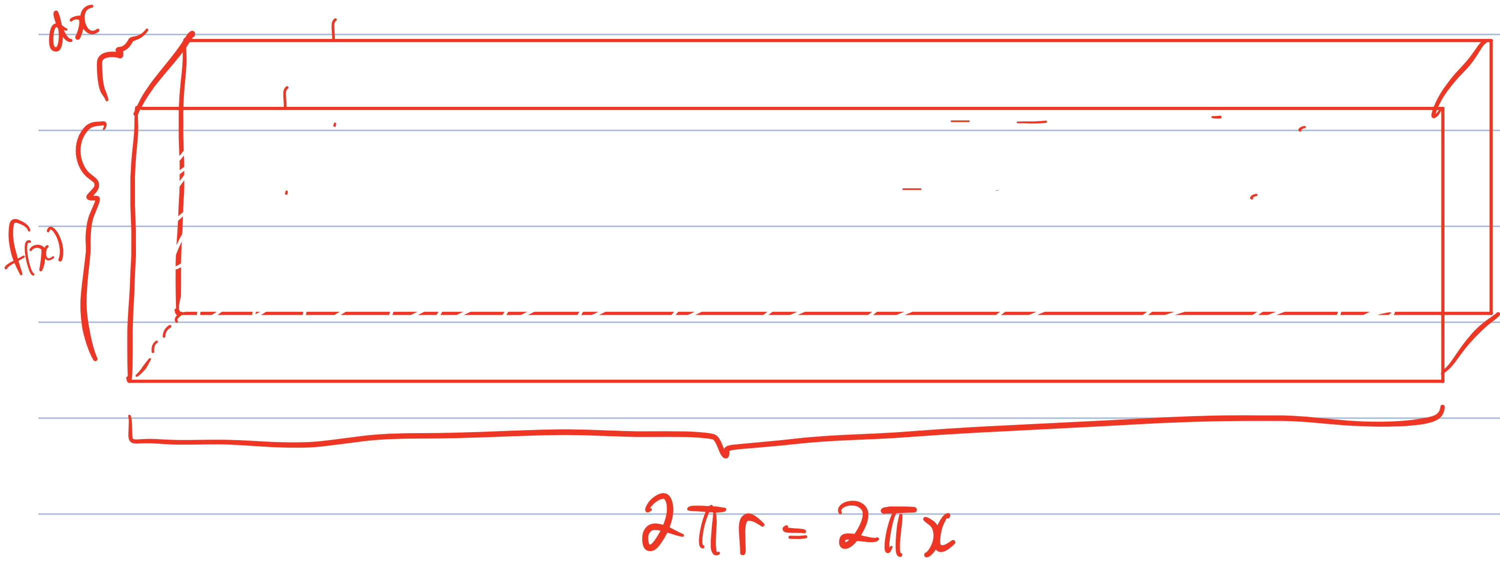
- circumference

  2π radius $= 2\pi x$

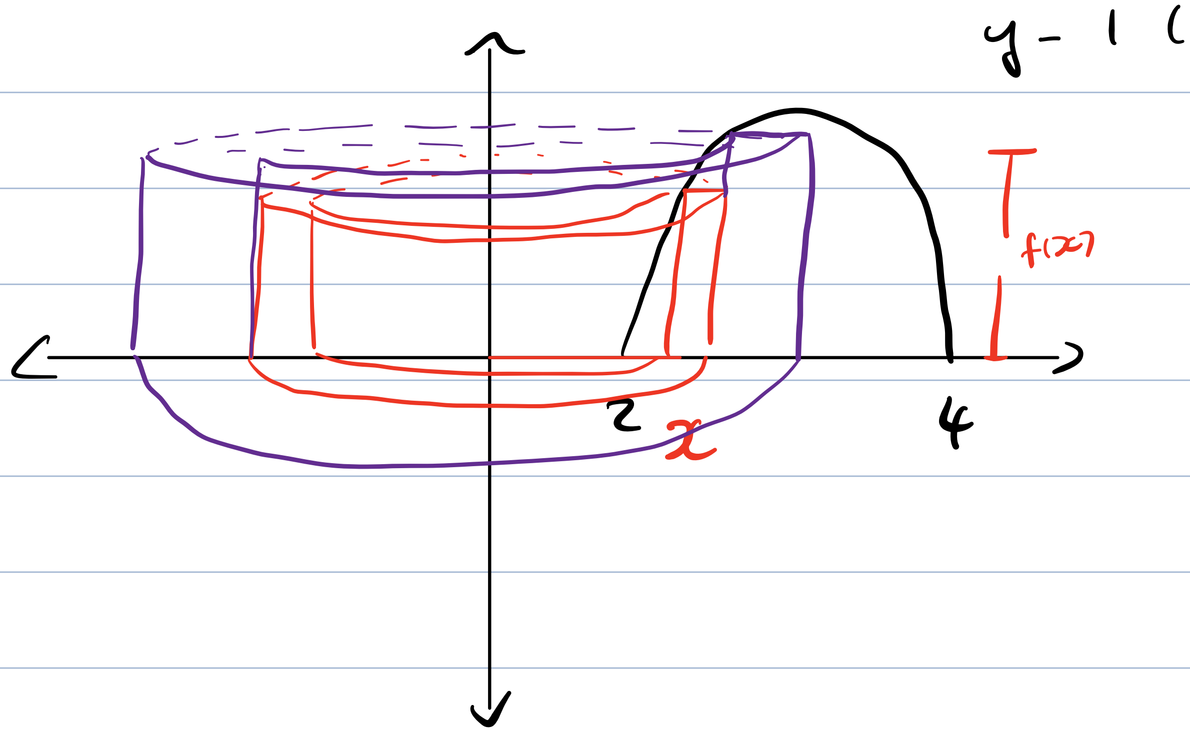
- Area circumf $\times$ height $= 2\pi x f(x)$

- volume : $(2\pi x)(f(x))\, dx$

  $= 2\pi f(x)$



dx
f(x)
2πr = 2πx



y - 1 (
f(x)
2
x
4

@

Example: Shell Method:

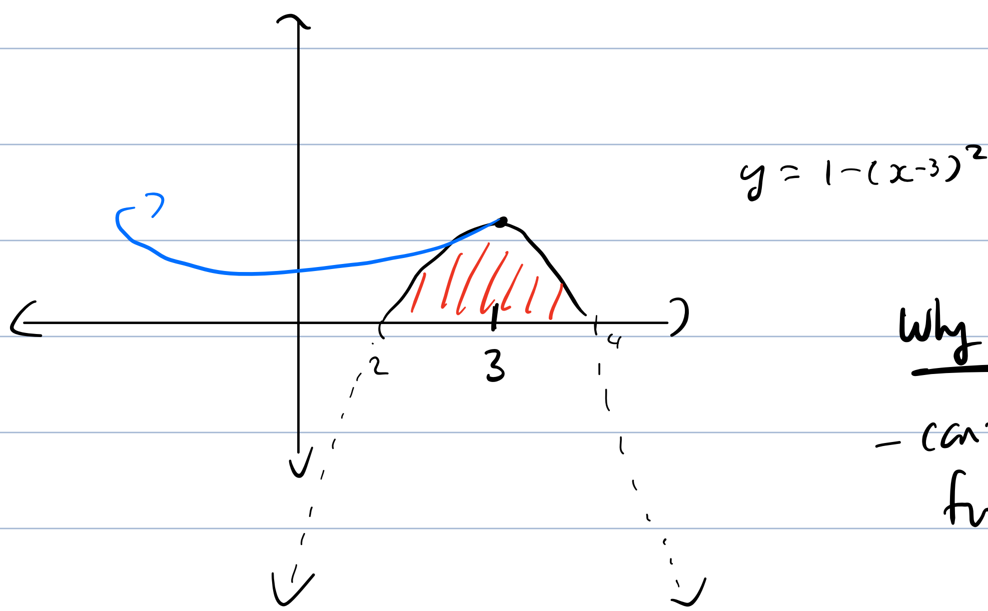
let R be region bounded by curves

$y = 1-(x-3)^2$ and $y=0$, and

V be volume formed by rotating R

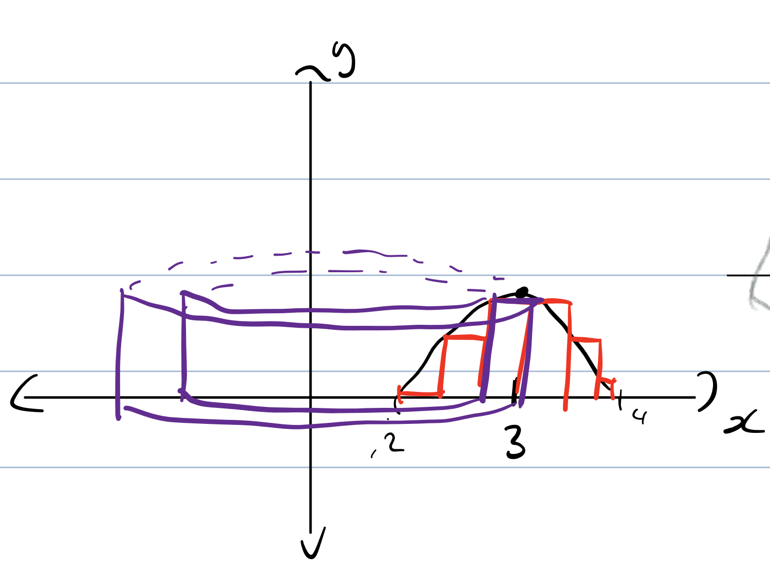
about y-axis.

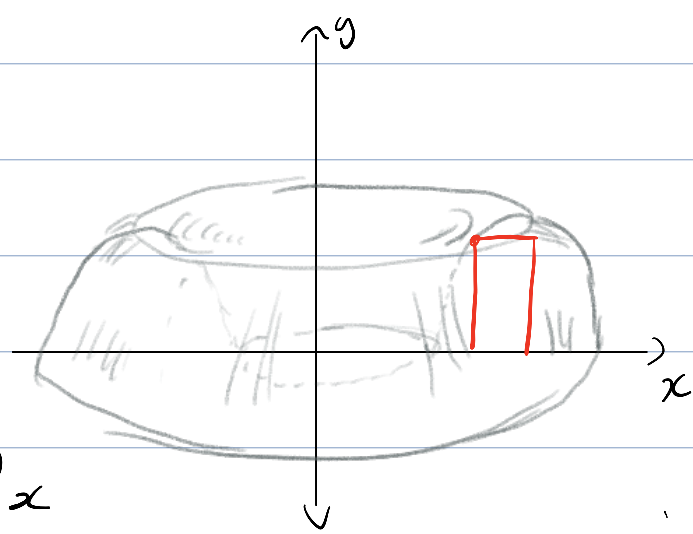
① Shell method:



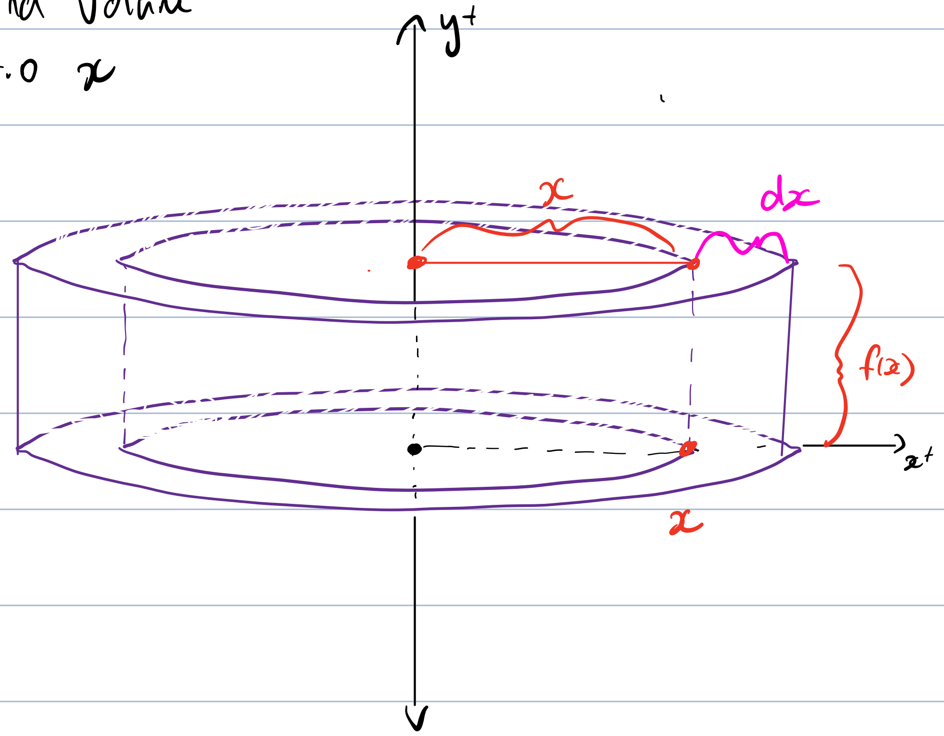
Why shell?

- can't express as function of y.





Find volume
i.t.o x



- Circumference (inside band)

$= 2\pi \text{ radius} = 2\pi x$

- inside band surface area

$=$ Circumference $\cdot$ height

$= 2\pi x \cdot f(x)$

- Volume $=$ area $\cdot$ depth $= 2\pi x\, f(x) \cdot dx$.

Now take limit as "$dx \to 0$" and

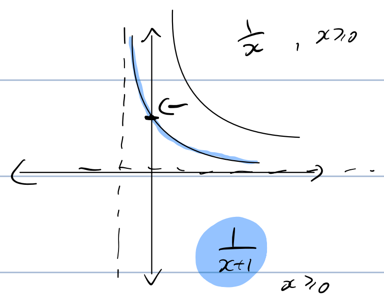
$$2\pi \int_2^4 x \cdot f(x)\, dx$$

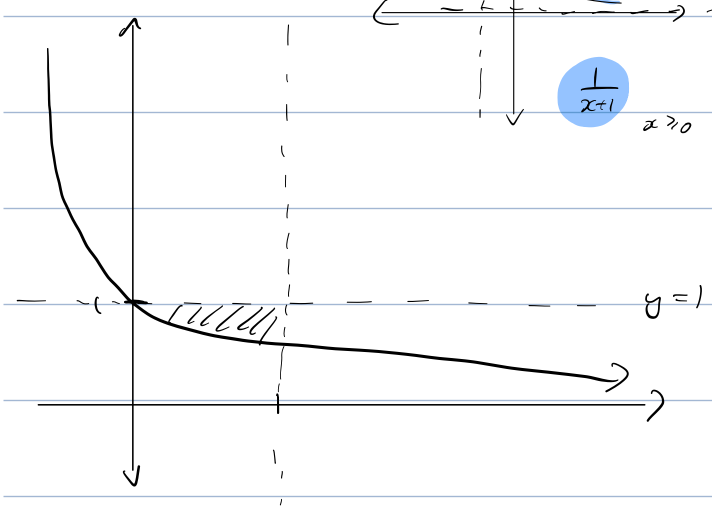
$$= 2\pi \int_2^4 x\left(1-(x-3)^2\right) dx$$

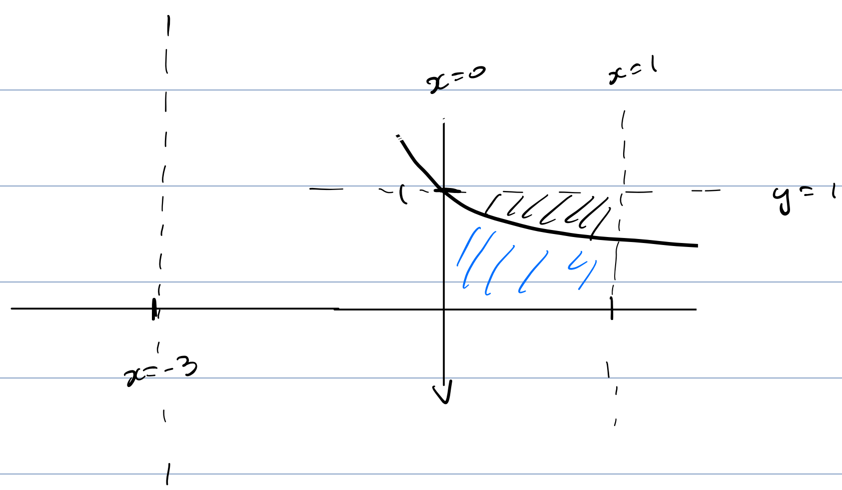
$$= 8\pi$$

Example: This example does not rotate about $x=0$, but $x=-3$.

- $y = \frac{1}{x+1}$, $x=1$, $y=1$ about $x=-3$







Idea:

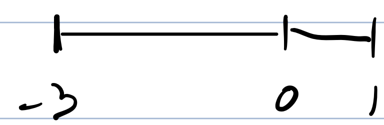
① Find area of //// part rotated, and subtract from big cylinder.

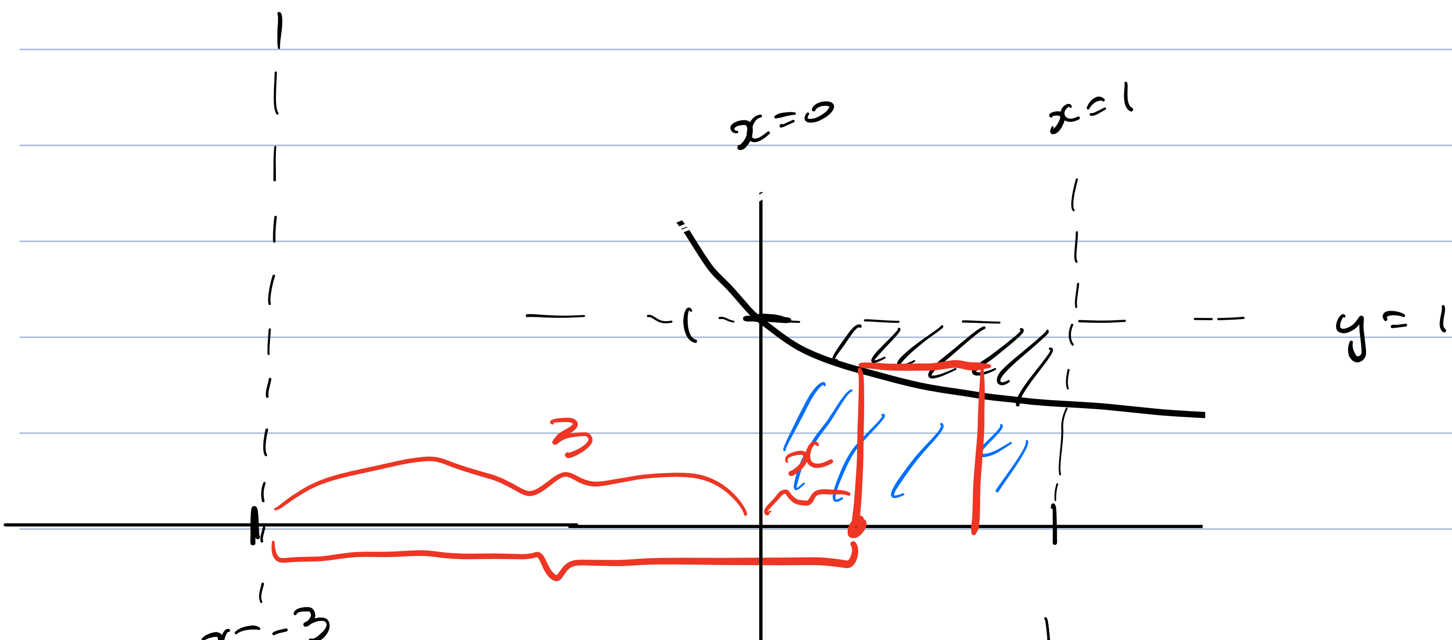
$$\left(\pi r_{outer}^2 - \pi_{inner}^2\right) \cdot \text{height}$$

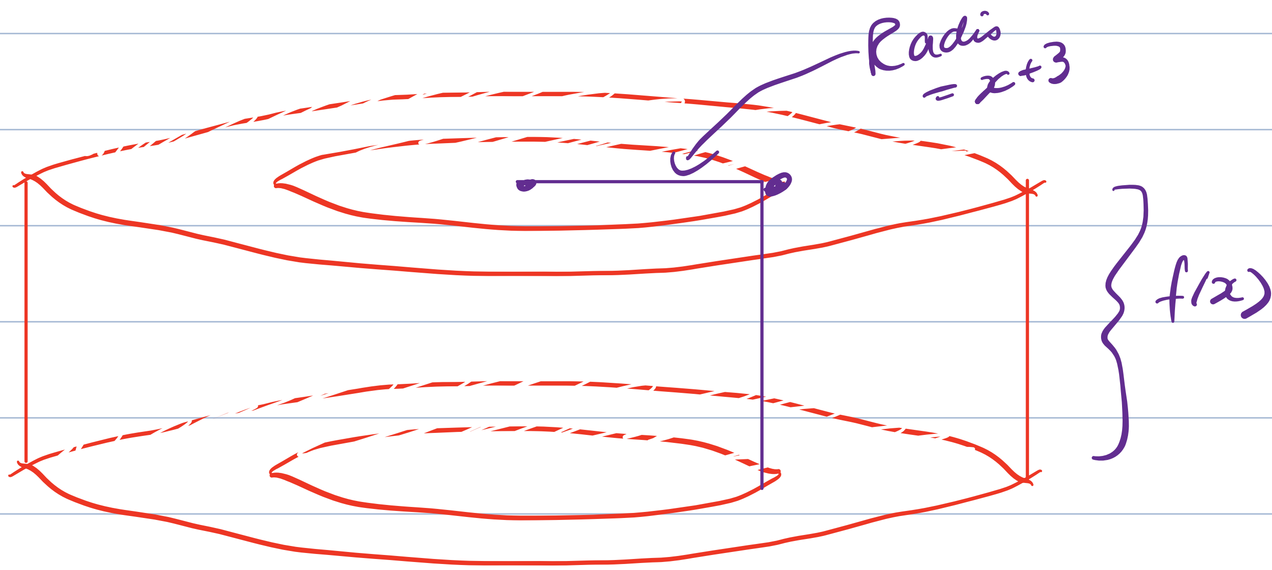
$$= \left(\pi 4^2 - \pi 9^2\right) \text{hed}$$

$$= \pi(16-9)$$

$$= \underline{7\pi} \quad \text{"big cylinder"}$$







- 2π (radis) $= 2\pi\,(x+3)$ circumf
- Area = circumfere · height $= 2\pi(x+3) \cdot f(x)$
- Vol = area · depth $= 2\pi(x+3)\, f(x)\, dx$.

So Area $= 2\pi \int_0^1 (x+3) f(x)\, dx$

$$= 2\pi \int_0^1 \frac{x+3}{x+1}\, dx$$

$$= 2\pi \int_0^1 \frac{x}{x+1}\, dx + 6\pi \int_0^1 \frac{1}{x+1}\, dx$$

$u = x+1$

$du = dx$

$x = u-1$

$$= 2\pi \int \frac{u-1}{u}\, du + 6\pi \ln|x+1| \Big|_0^1$$

$$= 2\pi \int 1 - \frac{1}{u}\, du + 6\pi\left(\ln(2) + \ln(1)\right)$$

$$= 2\pi \left(u - \ln|u| \right) \Big| + 6\pi \ln(2)$$

$$= 2\pi \left(x+1 - \ln(x+1) \Big|_0^1 \right) + 6\pi \ln(2)$$

$$= 2\pi \left(2 - \ln(2) - \left(1 - \ln(1) \right) \right) + 6\pi \ln(2)$$

$$= 2\pi \left(1 - \ln(2) \right) + 6\pi \ln(2)$$

$$= 2\pi - 2\pi \ln(2) + 6\pi \ln(2)$$

$$= 2\pi + 4\pi \ln(2)$$

Last step

- Find vol:

$$7\pi - 2\pi - 4\pi \ln(2)$$

$$= 5\pi - 4\pi \ln(2)$$

$$= \pi \left(5 - 4\ln(2) \right)$$