Discussion 5
Shell Method

Dec 6, 2024

Last Time:

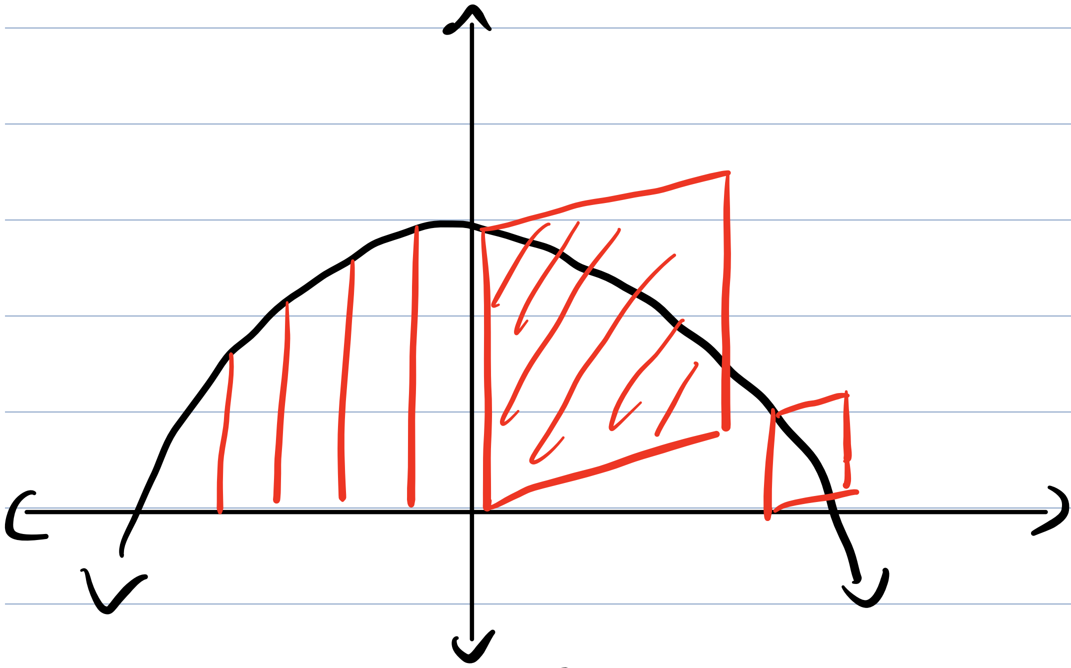
- Trigonometric substitution
- Integration by Parts

Today (Areas and volumes)

- Areas and volumes
- Disk Method
- Washer method
- Shell method
  - Geometric Idea behind Shell method
  - 2 Examples of using the shell method.

Areas and volumes

Example (Online hw 7)

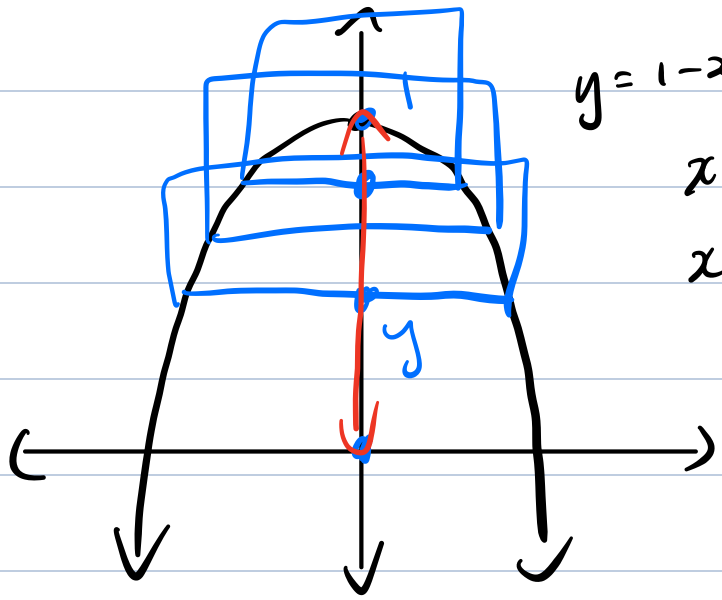


$$A(x) = (1-x^2)^2 = 1-2x^2+x^4$$

$$2\int_0^1 1-2x^2+x^4\,dx = 2\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1$$

$$= 2\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

Example (Online hw 7)



$y = 1 - x^2$

$x^2 = 1 - y$

$x = \sqrt{1-y}$.

$$\int_0^1 \left(2\sqrt{1-y}\right)^2 dy$$

$$= \int_0^1 4(1-y)\,dy$$

$$= \int_0^1 4 - 4y \, dy$$

$$= 4y - \frac{4y^2}{2} \Big|_0^1$$

$$= 4 - 4/2$$

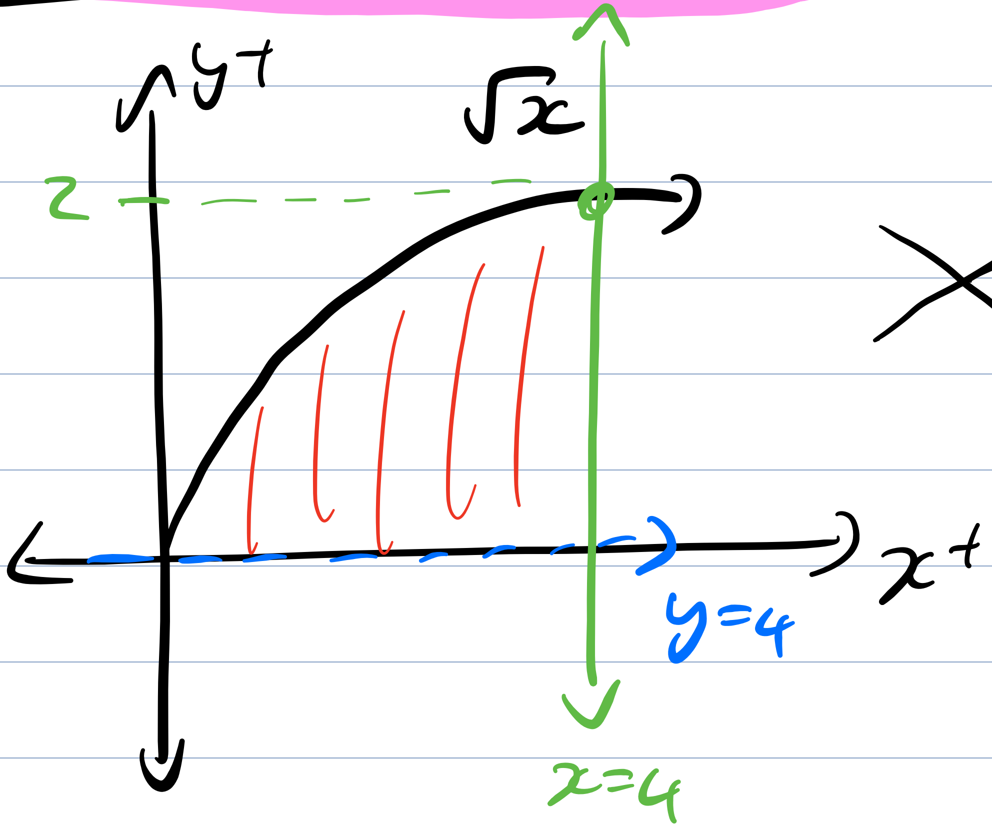
$$= 2.$$

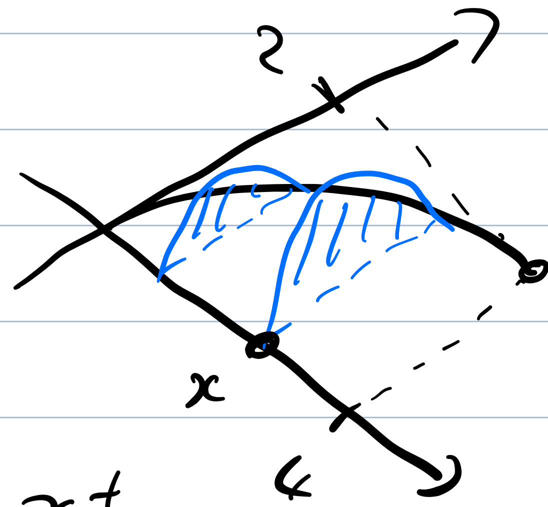
NB!

Need to sum along axis where $\perp$.

Example (Online hw 7)

Problem 8





- At point x, diameter of circle is $\sqrt{x}$.

$\Rightarrow$ radius at x is $r = \frac{\sqrt{x}}{2}$

- $A(x) = \frac{\pi r^2}{2}$

$$= \frac{\pi \left(\frac{\sqrt{x}}{2}\right)^2}{2} = \frac{\pi x}{8}$$

$$\int_0^4 A(x)\,dx = \int_0^4 \frac{\pi x}{8}\,dx = \frac{\pi x^2}{16}\Big|_0^4$$

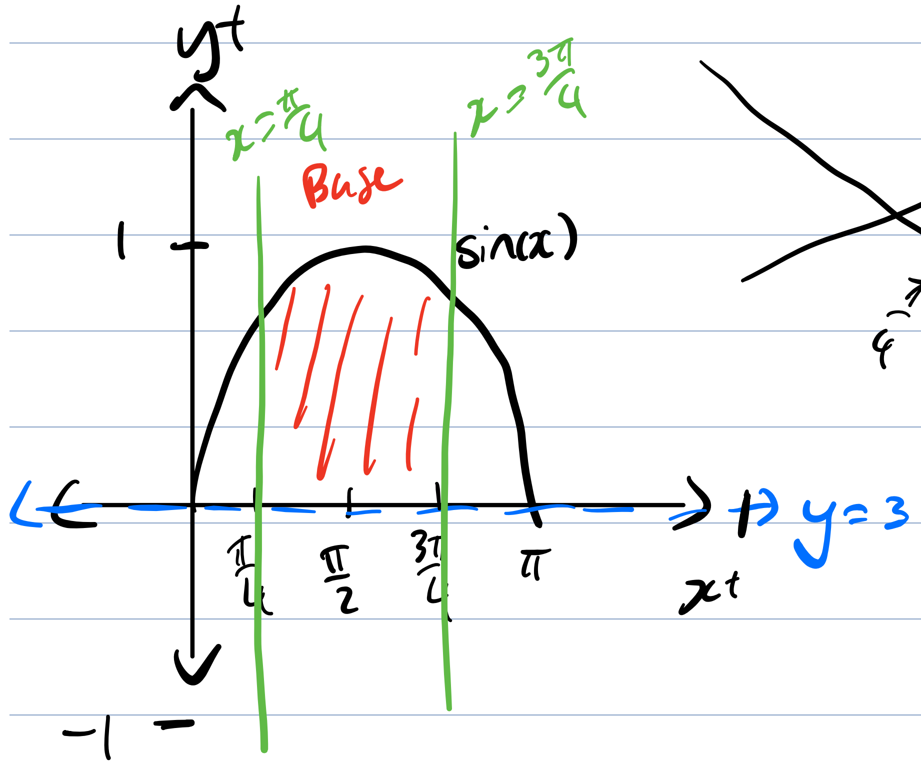
$$= \boxed{\pi}$$

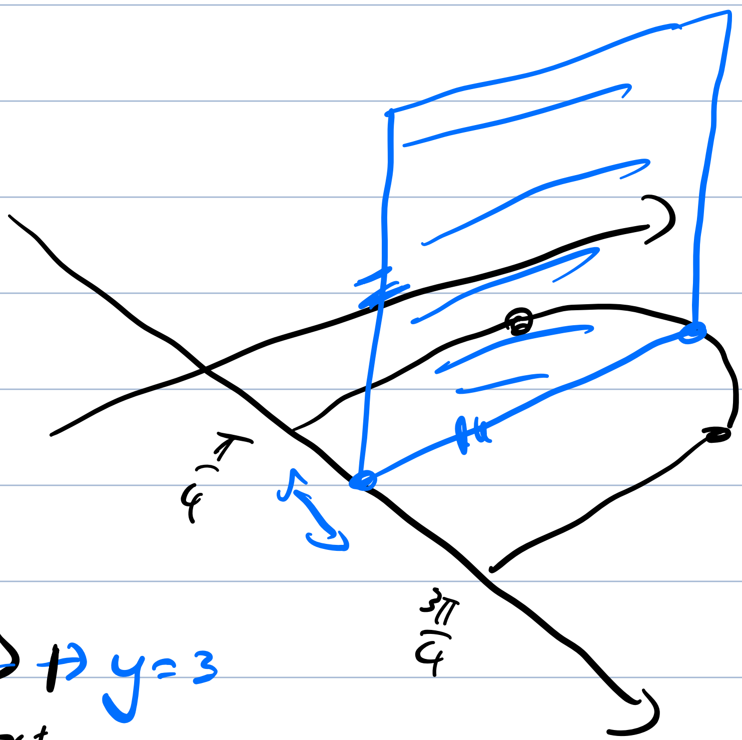
Example (Online hw 7)

Problem 9:

Square.

Base:





$A(x) := \sin^2(x)$.

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2(x)\,dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1-\cos^2(x)\,dx$$

$$\frac{d}{dx}\frac{\sin(2x)}{4} = \frac{\cos(2x)\cdot 2}{4}$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} dx - \left[\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}\,dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\cos(2x)}{2}\,dx\right]$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}\,dx - \left(\frac{\sin(2x)}{4}\right)\Bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

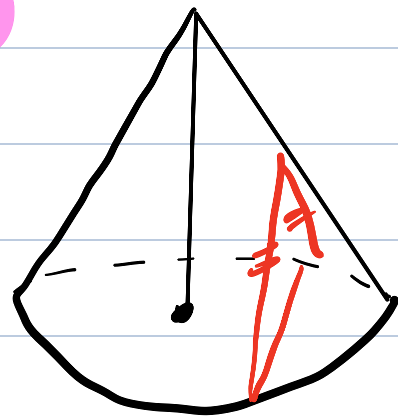
$$= \frac{1}{2}\left(\frac{3\pi}{4}-\frac{\pi}{4}\right) - \frac{1}{4}\left(\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right)$$

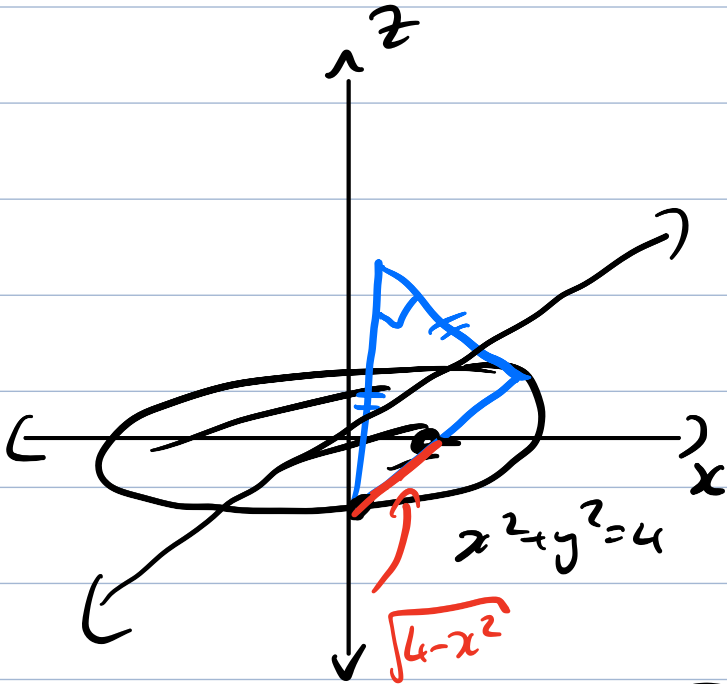
$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{4}(-1-1)$$

$$= \boxed{\frac{\pi}{4} + \frac{1}{2}.}$$

Example (Online hw 7)

Problem 10:





is osceles triangle
- two side equal length.

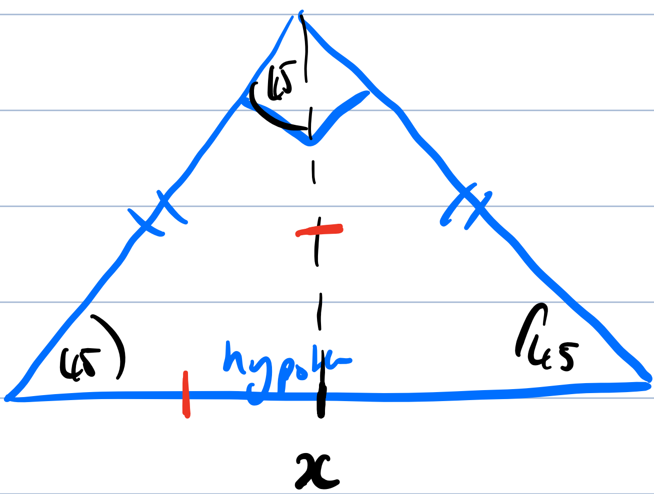
Fix $-2 \le x \le 2$:

What is the area $A(x)$?

$\frac{1}{2} bh$.

$A(x) = \frac{1}{2}\left(\overbrace{2\sqrt{4-x^2}}^{b} \cdot \overbrace{\sqrt{4-x^2}}^{h}\right)$

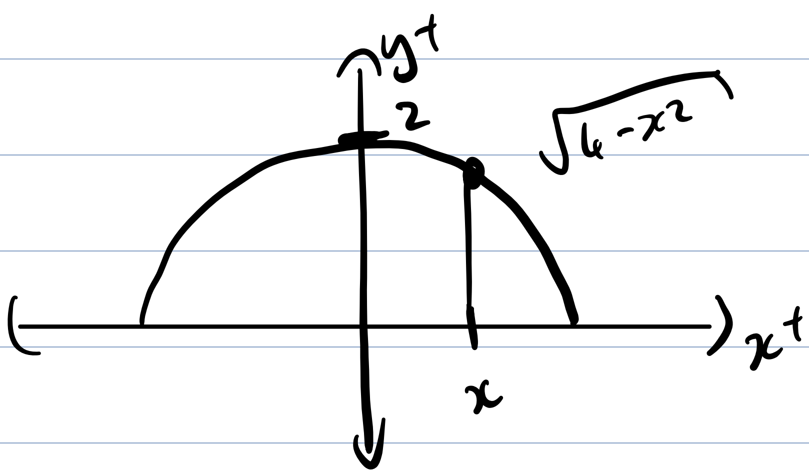
$= 4-x^2$



$\int_{-2}^{2} A(x)\,dx$

$= \int_{-2}^{2} 4-x^2\,dx$

$= 4x - \frac{x^3}{3}\Big|_{-2}^{2}$



$= 4(2) - \frac{8}{3} - \left(-8 - \left(\frac{-8}{3}\right)\right)$

$= 8 - 8/3 + 8 - 8/3$

$= 16 - \frac{16}{3}$.

1.) Disk Method

See lecture notes for examples.

2.) Washer method

See lecture notes

for examples.

3. Shell Method:

Why Shell method?

- The shell method is used when you are given curves with respect to one variable, that cannot easily be written explicitly i.t.o. other.

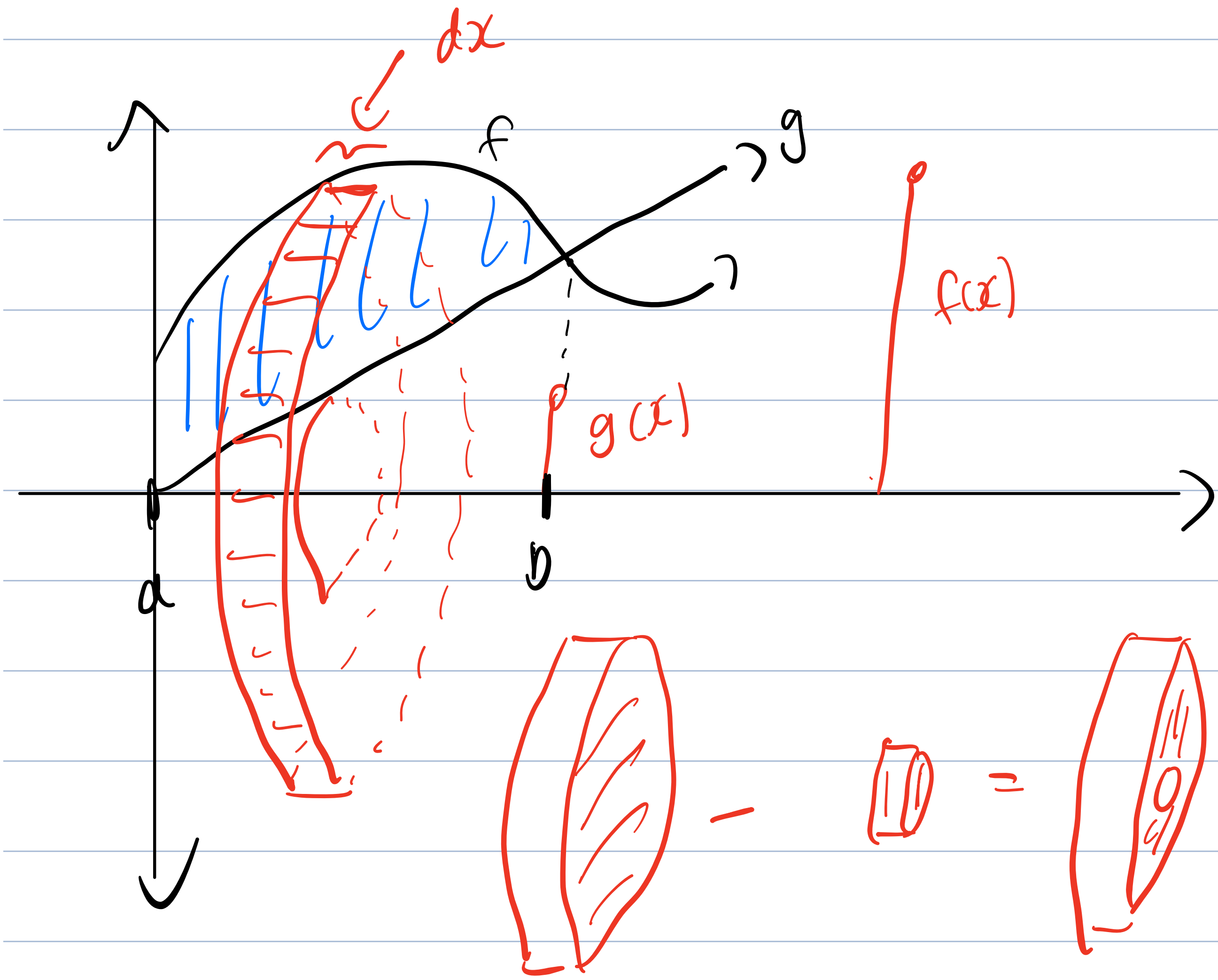
Example: $y = (x-3)^2(x-1)$ about the y-axis. If you wanted to use the washer method you need to write $x =$ "just y's".

Shell vs Washer

Washer Method
- Variable of integration and axis of rotation is the same.

Shell Method
- Variable of integration and axis of rotation differ.

Recall washer method.



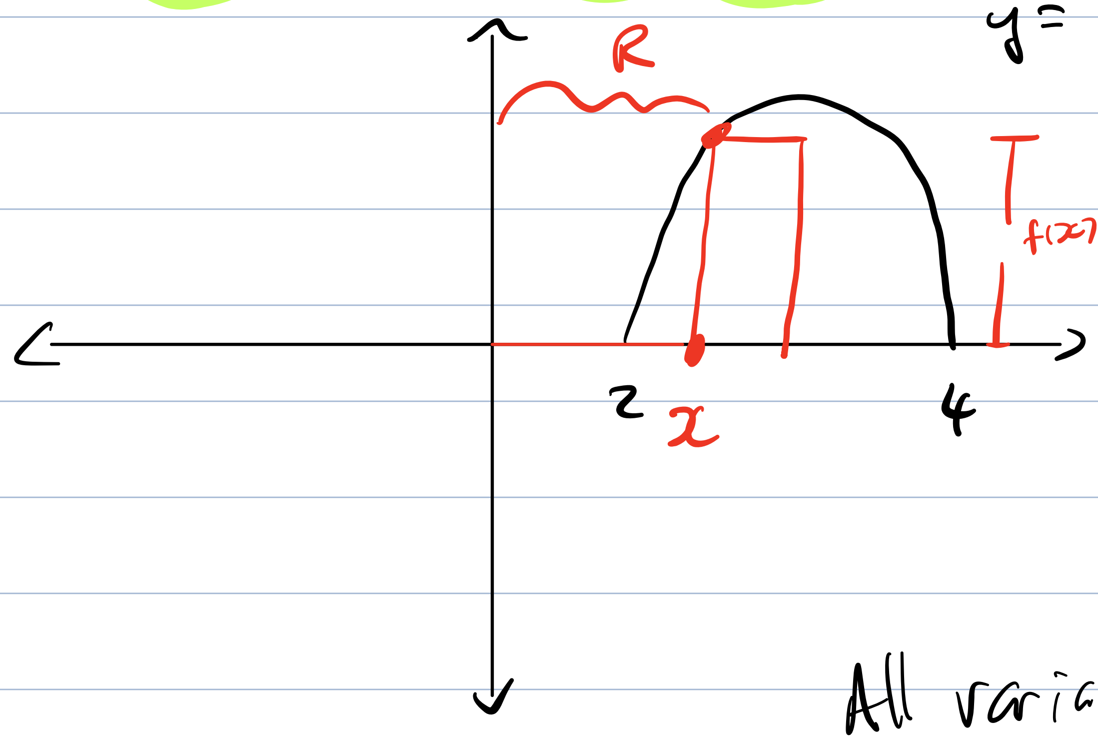
Cyllinder volume:

$$= \pi r_{outer}^2 \Delta x - \pi r_{inner}^2 \Delta x$$

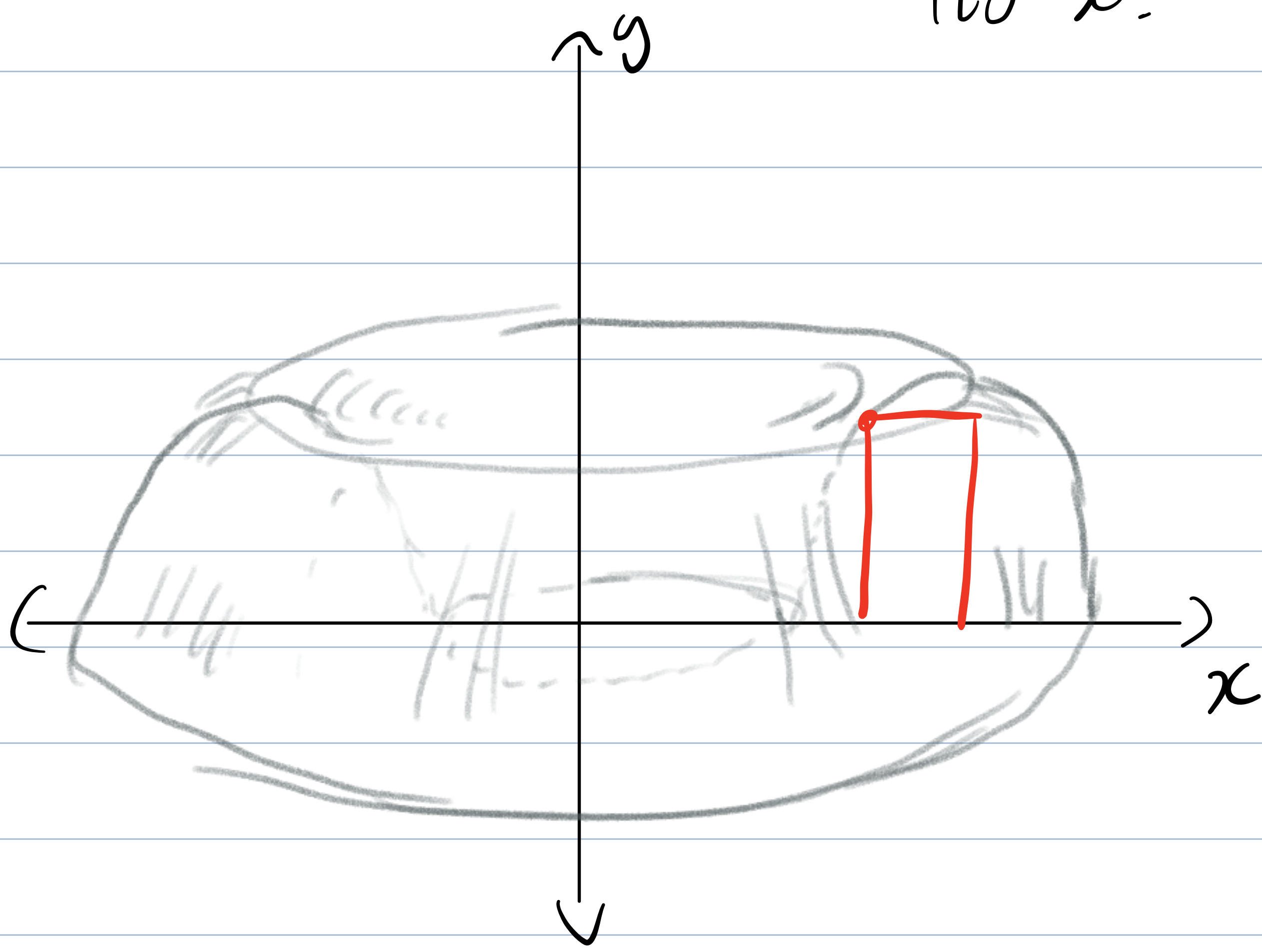
$$= \pi f(x)^2 \Delta x - \pi g(x)^2 \Delta x$$

Explaining The shell method:

$y = 1-(x-3)^2$

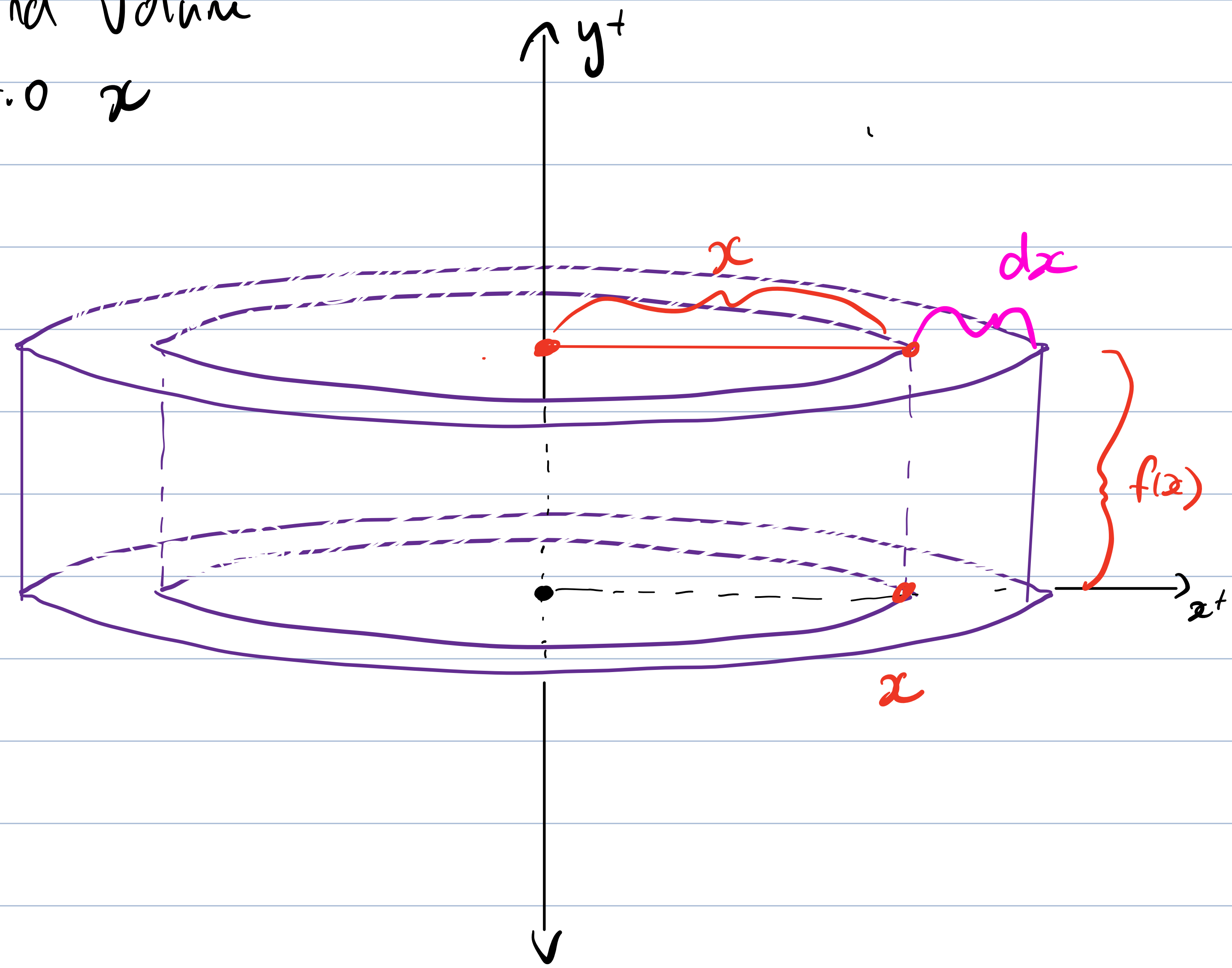


All variables ito x!



Find volume

i.t.o x



How to find volume of cylinder?

① outside surface area:

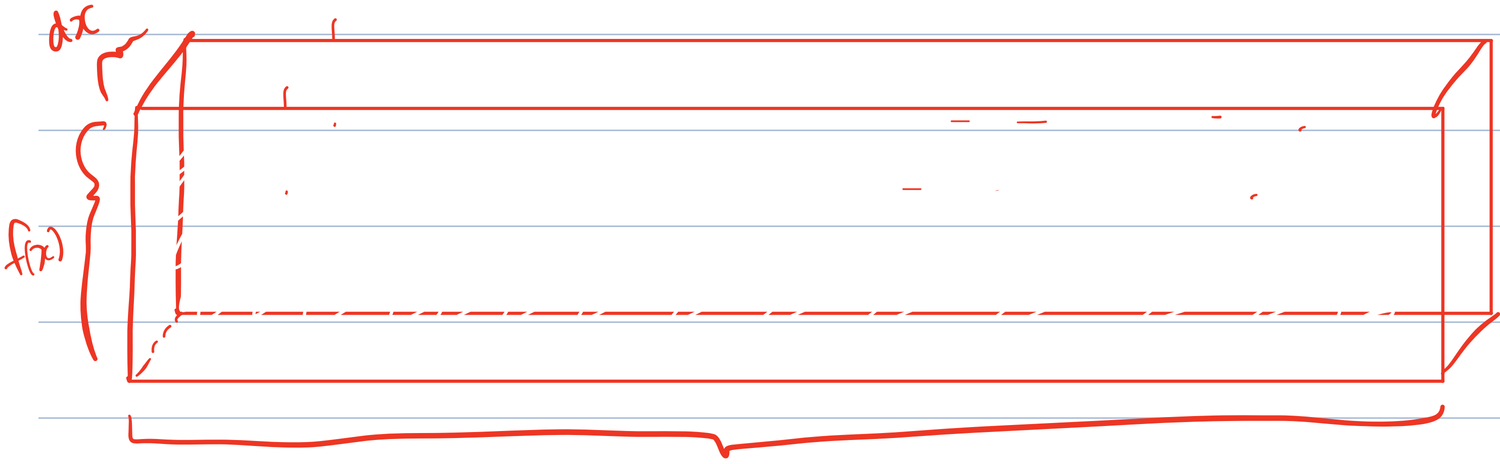
- circumference

  2π radius $= 2\pi x$

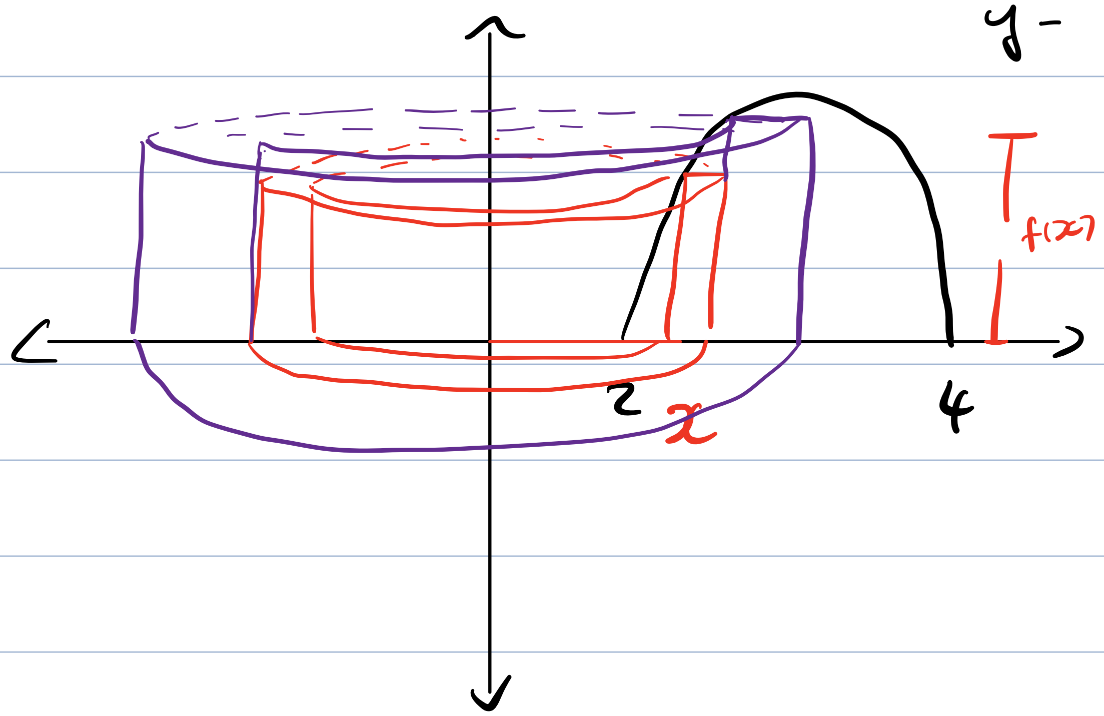
- Area circumf $\times$ height $= \underline{2\pi x\, f(x)}$

- volume : $(2\pi x)(f(x))\,dx$

  $= 2\pi f(x)$



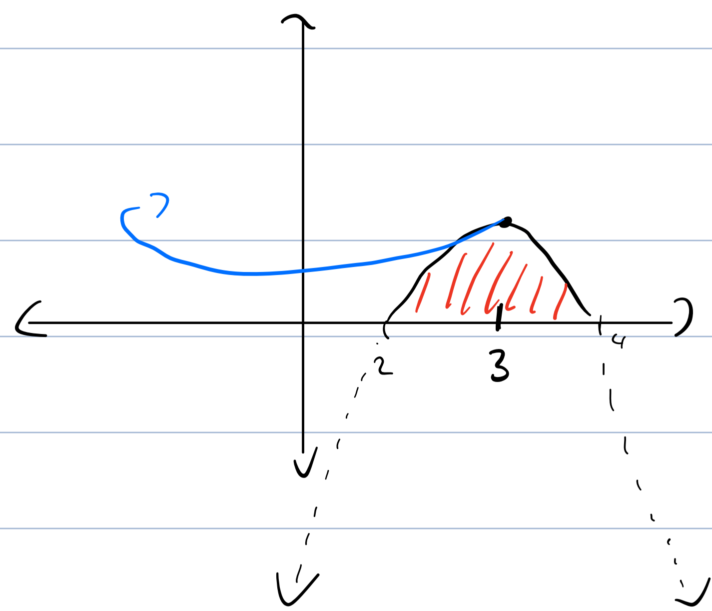
$2\pi r = 2\pi x$



Example : Shell Method :

let R be region bounded by curves
$y = 1-(x-3)^2$ and $y=0$, and
V be volume formed by rotating R
about y-axis.

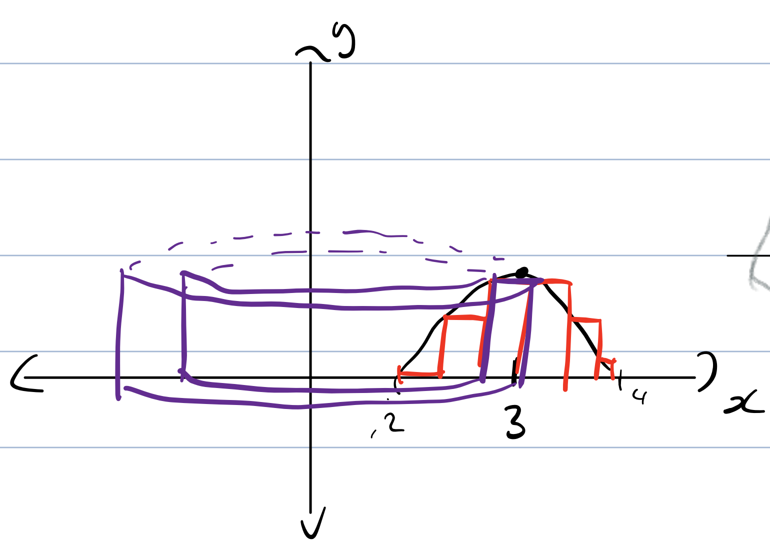
① Shell method:

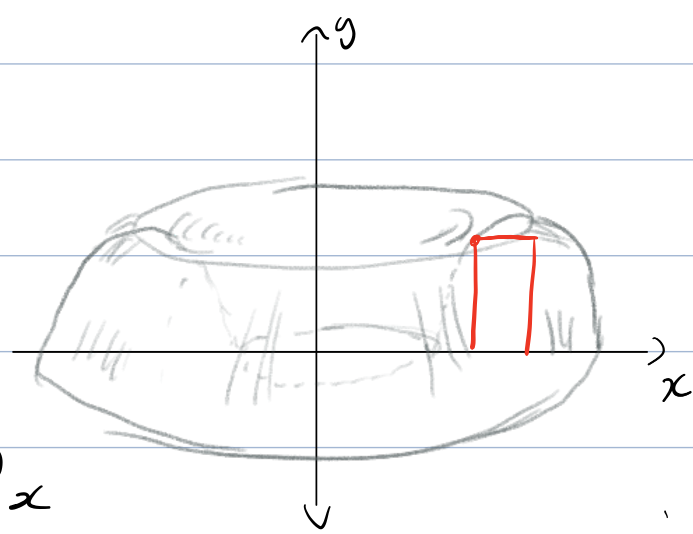


$y = 1-(x-3)^2$

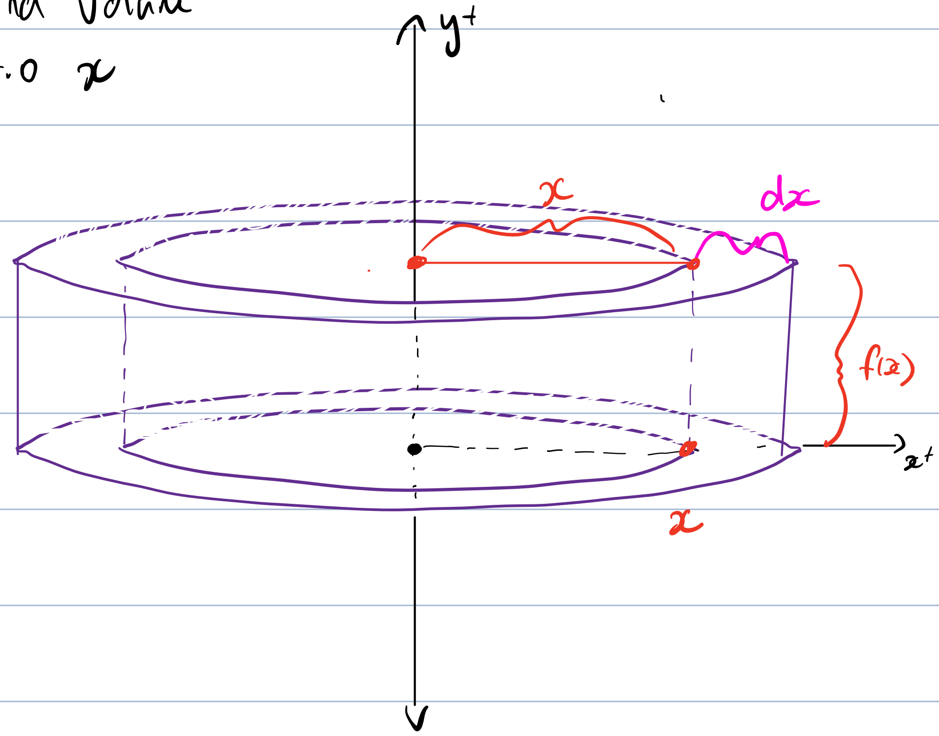
Why shell?

- can't express as function of y.





Find volume
i.t.o x



- Circumference (inside band)

$= 2\pi$ radius $= 2\pi x$

- inside band surface area

= Circumference · height

$= 2\pi x \cdot f(x)$

- Volume = area · depth $= 2\pi x\, f(x) \cdot dx$.

Now take limit as "$dx \to 0$" and

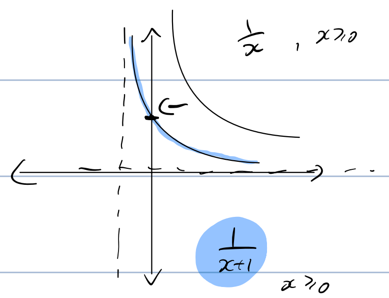
$$2\pi \int_2^4 x \cdot f(x)\, dx$$

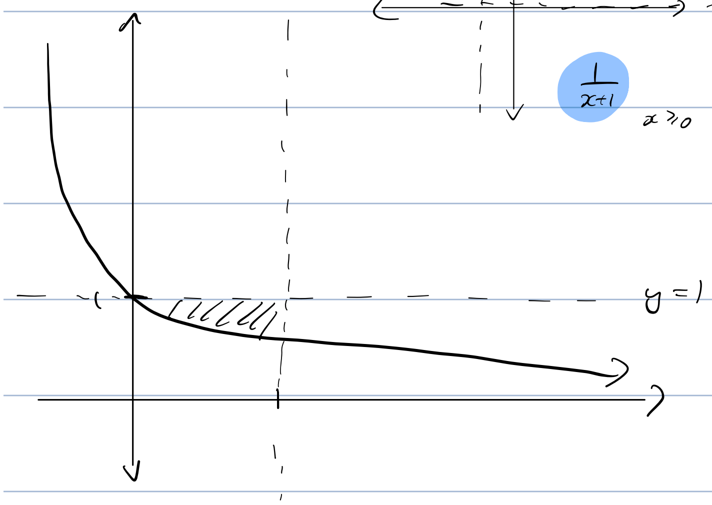
$$= 2\pi \int_2^4 x\left(1-(x-3)^2\right) dx$$

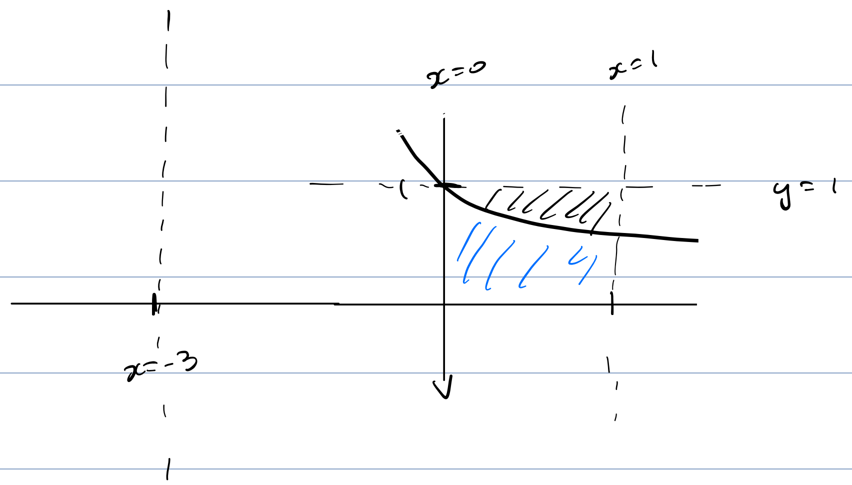
$$= 8\pi$$

Example: This example does not rotate about $x=0$, but $x=-3$.

- $y = \frac{1}{x+1}$, $x=1$, $y=1$ about $x=-3$







Idea:

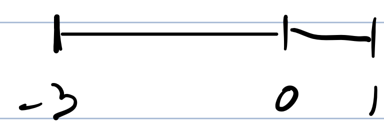
① Find area of //// part rotated, and subtract from big cylinder.

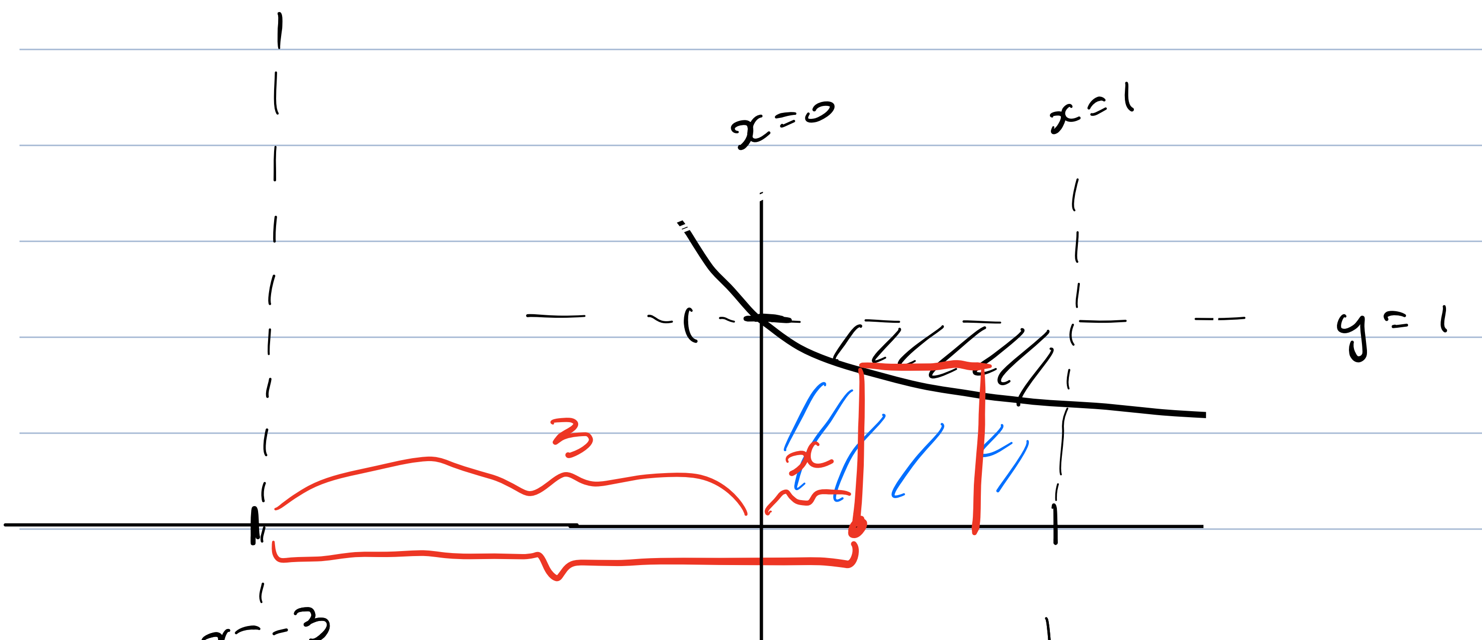
$$\left(\pi r_{outer}^2 - \pi r_{inner}^2\right) \cdot \text{height}$$

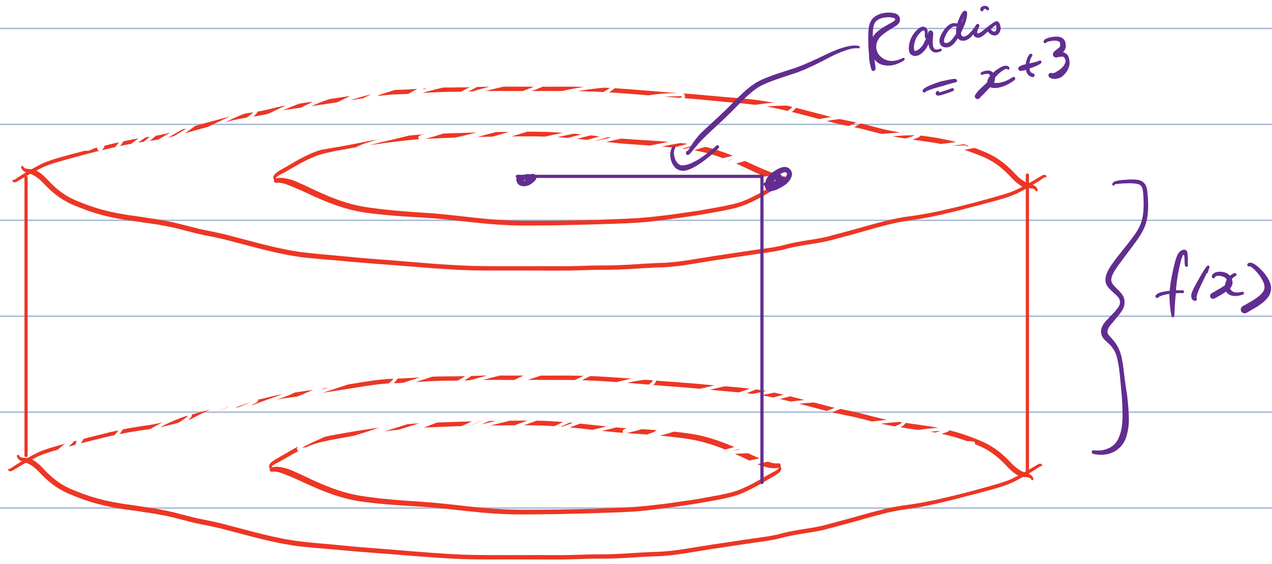
$$= \left(\pi 4^2 - \pi 9^2\right) \text{ht}$$

$$= \pi(16 - 9)$$

$$= 7\pi$$ "big cylinder"







- $2\pi(\text{radius}) = 2\pi(x+3)$ circumf
- Area = circumference · height = $2\pi(x+3)\cdot f(x)$
- Vol = area · depth = $2\pi(x+3)\, f(x)\, dx$.

So Area $= 2\pi \int_0^1 (x+3)\, f(x)\, dx$

$$= 2\pi \int_0^1 \frac{x+3}{x+1}\, dx$$

$$= 2\pi \int_0^1 \frac{x}{x+1}\, dx + 6\pi \int_0^1 \frac{1}{x+1}\, dx$$

$u = x+1$

$du = dx$

$x = u-1$

$$= 2\pi \int \frac{u-1}{u}\, du + 6\pi \ln|x+1| \Big|_0^1$$

$$= 2\pi \int 1 - \frac{1}{u}\, du + 6\pi\left(\ln(2) + \ln(1)\right)$$

$$= 2\pi \left(u - \ln|u| \right) \Big| + 6\pi \ln(2)$$

$$= 2\pi \left(x+1 - \ln(x+1) \Big|_0^1 \right) + 6\pi \ln(2)$$

$$= 2\pi \left(2 - \ln(2) - \left(1 - \ln(1) \right) \right) + 6\pi \ln(2)$$

$$= 2\pi \left(1 - \ln(2) \right) + 6\pi \ln(2)$$

$$= 2\pi - 2\pi \ln(2) + 6\pi \ln(2)$$

$$= 2\pi + 4\pi \ln(2)$$

Last step

- Find vol:

$$7\pi - 2\pi - 4\pi \ln(2)$$

$$= 5\pi - 4\pi \ln(2)$$

$$= \pi \left(5 - 4\ln(2) \right)$$