Discussion Notes

Sept 24, 2024

Improper Integrals

Last time

- Trigonometric substitution
- Integration by Parts

Today

- Improper Integrals

Improper Integration:

Recall the Fundamental Theorem of Calculus:

The Fundamental Theorem of Calculus is a statement about the inverse relationship between differentiation and integration. It comes in two parts, depending on whether we are differentiating an integral or integrating a derivative. Under suitable hypotheses on the functions f and F, the Fundamental Theorem of Calculus states that

(i) $\int_a^b F'(x)\,dx = F(b) - F(a)$ and

(ii) if $G(x) = \int_a^x f(t)\,dt$, then $G'(x) = f(x)$.

More precisely:

The result is stated in two parts. The first is a computational statement that describes how an antiderivative can be used to evaluate an integral over a particular interval. The second statement is more theoretical in nature, expressing the fact that every continuous function is the derivative of its indefinite integral.

Theorem 7.5.1 (Fundamental Theorem of Calculus). (i) *If $f : [a, b] \to \mathbf{R}$ is integrable, and $F : [a, b] \to \mathbf{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f = F(b) - F(a).$$

(Integrating a derivative)

(ii) *Let $g : [a, b] \to \mathbf{R}$ be integrable, and define*

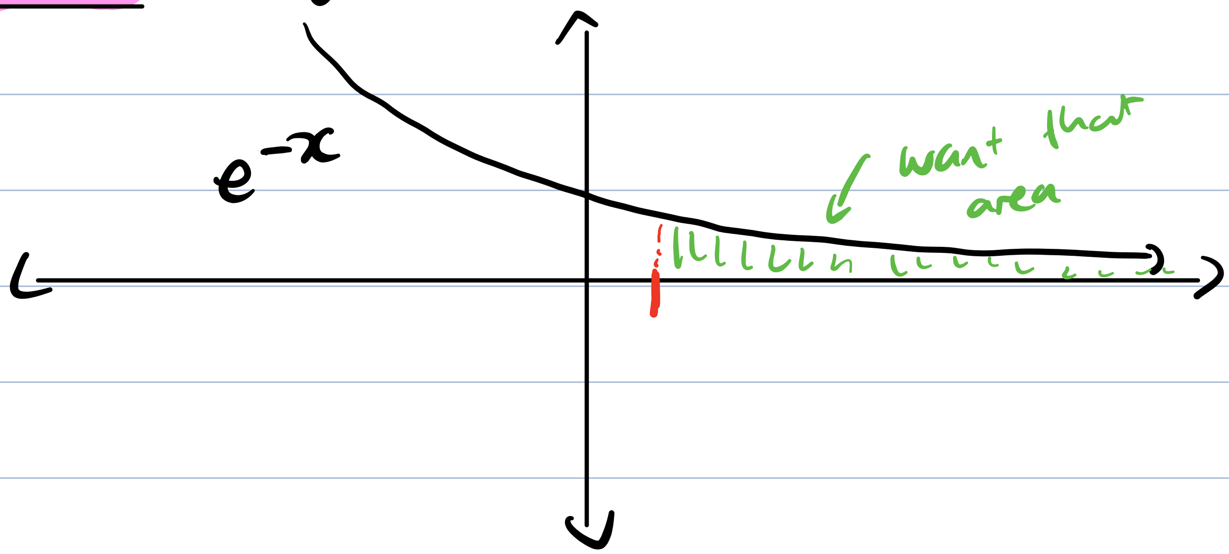
$$G(x) = \int_a^x g$$

(Differentiating an integral)

for all $x \in [a, b]$. Then, G is continuous on $[a, b]$. If g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

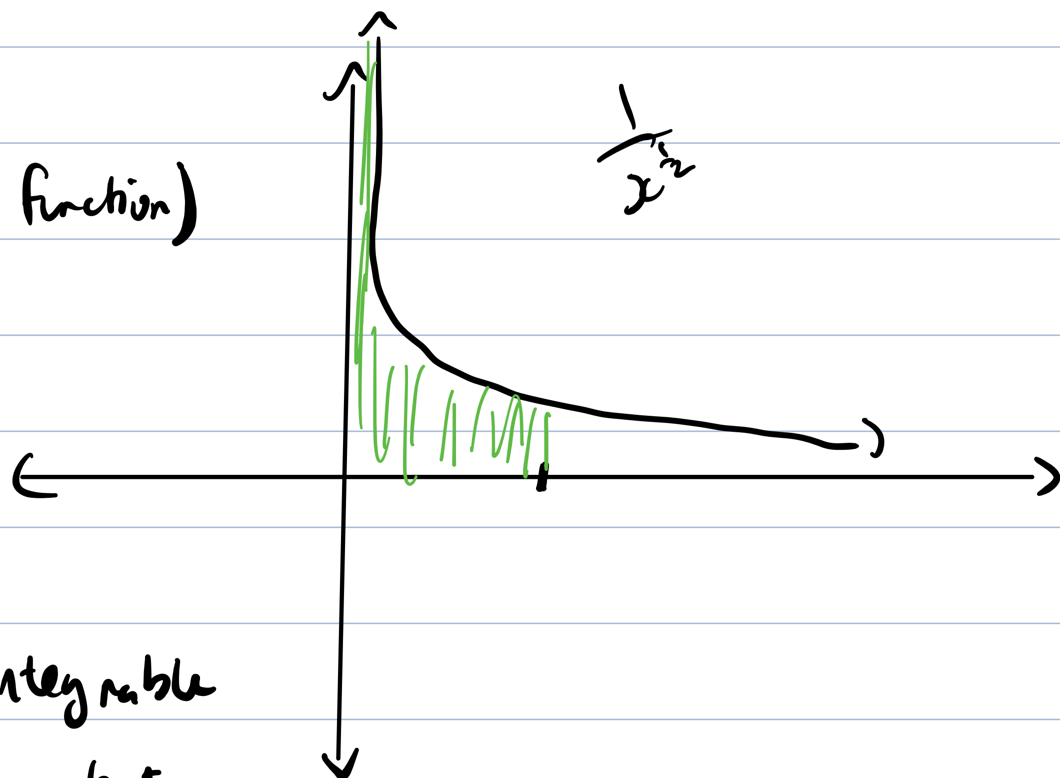
Why do we need Improper Integrals:

Example 1 (Integrals over unbounded intervals)



① we want integrals over $[a,\infty)$ not just over bounded closed and bounded one

Example 2:
(non-integrable function)



$\frac{1}{x^{1/2}}$ not integrable over $[0,1]$, but if the area under the curve is finite, we want to be able to calculate it.

Solution :

(1) $$\underbrace{\int_a^b f(x)\,dx}_{\text{not FTC}} = \underbrace{\lim_{\varepsilon \to 0^+} \overbrace{\int_{a+\varepsilon}^b f(x)\,dx}^{\text{FTC}}}_{\text{Then take limit}} \overset{\text{Hopefully}}{=} \int \lim_{\varepsilon \to 0^+} f \mathbb{1}_{[a+\varepsilon, b]}$$

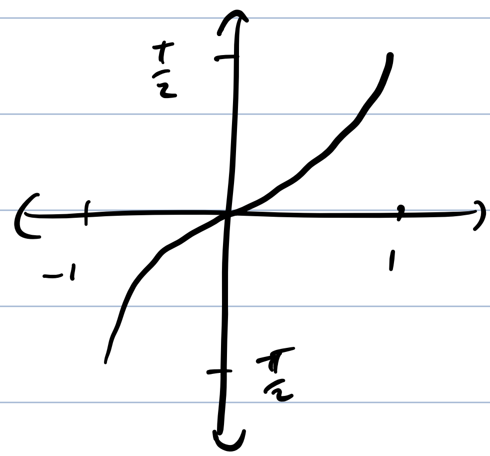
Example 1

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

- $\displaystyle\lim_{x\to 1^-} \frac{1}{\sqrt{1-x^2}} = \infty$

So vertical asymptote.

$$\lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}}\,dx$$



$$\lim_{t\to 1^-} \arcsin(x)\Big|_0^t$$

$$= \arcsin(1) - 0 = \frac{\pi}{2}$$

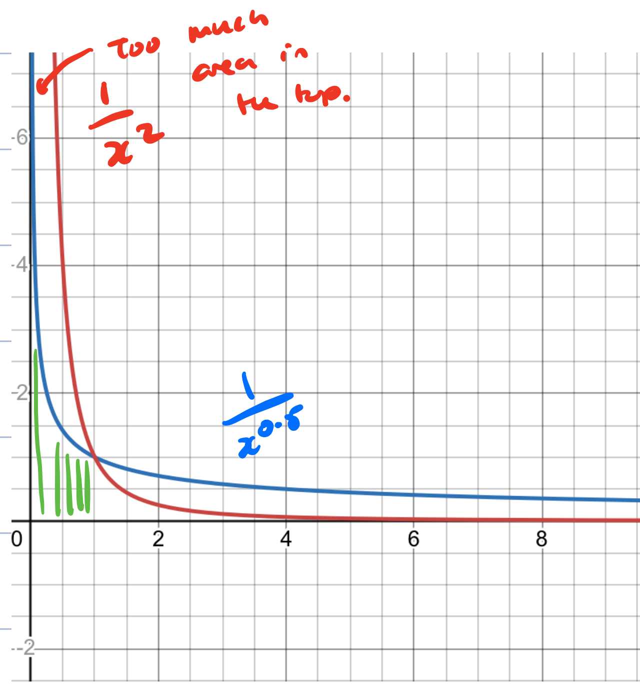
Extra Problems:

1. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ (solution: $1/2$)
2. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$ (solution: $\frac{\pi}{3\sqrt{3}}$)
3. $\displaystyle\int_0^\infty \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$ (solution: ∞)
4. $\displaystyle\int_0^{\pi/4} \left(\csc^2(\theta)-\csc(\theta)\cot(\theta)\right)d\theta$ (solution: $\sqrt{2}-1$)
5. $\displaystyle\int_{-1}^0 \frac{1}{x^2+2x+1}\,dx$ (solution: ∞)
6. Given that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, evaluate $\displaystyle\int_0^\infty x^2e^{-x^2}\,dx$. (solution: $\frac{\sqrt{\pi}}{4}$)

Example 2 :

$$\int_0^1 \frac{1}{x^p}\,dx$$

$$= \lim_{t\to 0^+}\int_t^1 x^{-p}\,dx$$



If $p=1$, $\int x^{-p}\,dx = \ln|x|$

$$\lim_{t\to 0^+} \ln|x|\Big|_t^1 = \cancel{\ln|1|}^{\,0} - \lim_{t\to 0^+}\ln|t| = -\infty$$

Problem! Integral DNE

If $p>1$, $\int x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}$

Then $\lim_{t\to 0^+}\int_t^1 x^{-p}\,dx = \lim_{t\to 0^+} \frac{x^{-p+1}}{-p+1}\Big|_t^1$

$$= \frac{1}{-p+1} - \lim_{t\to 0^+}\frac{x^{-p+1}}{-p+1}$$

since $p>1$, $-p+1<0$, $p-1>0$.

$$= \frac{-1}{-p+1} - \frac{1}{-p+1}\underbrace{\lim_{t\to 0^+}\frac{1}{x^{1-p}}} = \infty$$

If $p<1$,

same calculation as $p>1$

Then $\lim_{t\to 0^+}\int_t^1 x^{-p}\,dx = \frac{1}{-p+1} - \lim_{t\to 0^+}\frac{x^{-p+1}}{-p+1}$

since $p<1$, $-p+1>0$

so $\lim_{t\to 0^+} x^{-p+1} = 0$.

$$= \frac{1}{-p+1}$$

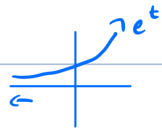
Example: Calculate

$$\int_{-\infty}^{1} x e^{x}\, dx .$$

$$(*) = \lim_{t\to-\infty} \left. xe^{x} - e^{x} \right|_{t}^{0}$$

$$= (0-1) - \left[\lim_{t\to-\infty} te^{t} - e^{t}\right]$$

$$= \boxed{-1} - \underbrace{\cancel{\lim_{t\to-\infty} te^{t}}}_{A}{}^{\nearrow 0} - \underbrace{\cancel{\lim_{t\to-\infty} e^{t}}}_{B}{}^{\nearrow 0}$$



see Below

$$A = \lim_{t\to-\infty} te^{t} = \lim_{t\to-\infty} \frac{t}{e^{-t}} \quad \leftarrow \left(\text{“}\tfrac{-\infty}{\infty}\text{” so L'Hosp Apply}\right)$$

$$\overset{L'H}{=} \lim_{t\to-\infty} \frac{1}{-te^{-t}}$$

$$= \lim_{t\to-\infty} \frac{1}{-t} \cdot e^{t} \quad \Big\} \text{ allowed since both limits exists.}$$

$$= \left(\lim_{t\to-\infty} \frac{1}{-t}\right)\left(\lim_{t\to-\infty} e^{t}\right)$$

$$= 0 \cdot 0$$

$$= 0$$

Thus

$$\int_{-\infty}^{0} x e^{x}\, dx = -1$$