Discussion Notes

October 8, 2024

Sequences and Series

Last Time

- Probability
- Work

Today

- Sequences
- Series

Ask Yourself:

- What is a sequence?
- What is a series?

Overview

- Algebraic Properties
- Geometric series (example)
- Telescopic Series (Example)
- Harmonic series (example)
- Test for divergence (Proposition)

Algebraic Properties

Thm: Let $(a_n) \to a$, $(b_n) \to b$, c constant.

1.) $(ca_n) \to ca$
2.) $(a_n + b_n) \to a+b$
3.) $a_n b_n \to ab$
4.) $\frac{a_n}{b_n} \to \frac{a}{b}$, provided $b \neq 0$
5.) $|a_n| \to |a|$

Order limit Thm

For sequences $(a_n) \to a$, $(b_n) \to b$
1.) If $a_n \geq 0$ for all n, $a \geq 0$
2.) If $a_n \geq b_n$ for all n, $a \geq b$.

Geometric Series :

- $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$

- $\displaystyle\sum_{n=0}^{\infty} x^n$ diverges if $|x| \geq 1$

so that we can divide by $x-1$

Observe that for $x \neq 1$, we have

$$(x-1)\left(1+x+x^2+\dots+x^N\right)$$

$$= x+x^2+\dots+x^{N+1}-1-x-\dots-x^N$$

$$= x^{N+1}-1$$

$$\Rightarrow \quad 1+x^2+\dots+x^N = \frac{x^{N+1}-1}{x-1}$$

Where

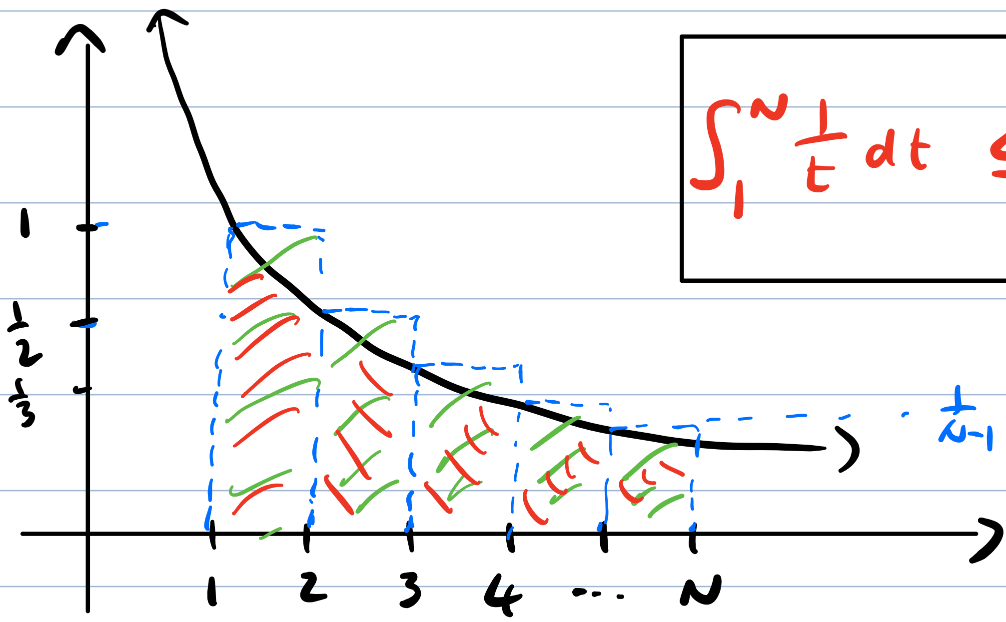
$$\lim_{N\to\infty} \frac{x^{N+1}-1}{x-1} = \begin{cases} \dfrac{1}{1-x} & |x|<1 \\ \infty & |x|>1 \end{cases}$$

And $1+x+x^2+\dots+x^N = N+1$ if $x=1$, so $\displaystyle\lim_{N\to\infty} \sum_{n=0}^{N} x^N = \infty$, and $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ oscillates between 0 and 2.

Harmonic Series :

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges, but very.}$$

• $S_N = \sum_{n=1}^{N} \frac{1}{n}$ with $\int_1^N \frac{1}{t}\, dt$.



$$\boxed{\int_1^N \frac{1}{t}\, dt \le \sum_{n=1}^{N-1} \frac{1}{n}}$$

But we know $\lim_{N\to\infty} \int_1^N \frac{1}{t}\, dt = \lim_{N\to\infty} \ln|t| \Big|_1^N$

$$= \lim_{N\to\infty} \ln|N| = \infty$$

So $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

P-series

$$\sum_{n=0}^{\infty} \left(\frac{1}{n}\right)^p = \begin{cases} \text{converge}, & p>1 \\ \text{diverge}, & p \le 1 \end{cases}$$

Telescopic Series

Given a sequence (b_n) we can form a series $\sum_{n=1}^{\infty}(b_n - b_{n+1})$.

Observe that the sequence of partial sums

$$S_N = (b_1 - \cancel{b_2}) + (\cancel{b_2} - \cancel{b_3}) + (\cancel{b_3} - b_4) + \cdots + (\cancel{b_N} - b_{N+1})$$

$$= b_1 - b_{N+1}.$$

So $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converge iff (b_n) converge.

Example: (convergence)

$$\sum_{n=1}^{\infty} \frac{2}{n^3+3n^2+2n}$$

well $\frac{2}{n(n^2+3n+2)} = \frac{2}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$

$2 = A(n^2+3n+2) + B(n^2+2n) + C(n^2+n)$

$2 = (A+B+C)n^2 + (3A+2B+C)n + 2A.$

$\Rightarrow \quad 2A = 2 \quad \Rightarrow \boxed{A=1}$

$\cdot \quad 1+B+C = 0$

$-(3+2B+C = 0)$

$-2-B \quad = 0$

$\boxed{B \quad = -2}$

$1-2+C = 0$

$\Rightarrow \boxed{C = 1}$

So $\sum_{n=1}^{\infty} \frac{1}{n} + \frac{1}{n+2} - \frac{2}{n+1}$

$= \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{b_n} - \underbrace{\left(\frac{1}{n+1} - \frac{1}{n+2}\right)}_{b_{n+1}.}$

And $\lim_{n\to\infty} S_n = \lim_{n\to\infty} b_1 - \lim_{n\to\infty} \cancel{b_{n+1}} \to 0$

$= b_1$

$= \boxed{\frac{1}{2}}$

- If the series converge, then the tail of the sequence of coefficients gets small.

Thm: If $\sum_{n\geq 1}^{\infty} a_n$ converge, then $(a_n) \to 0$.

Proof: Note that

$$S_N - S_{N-1} = \sum_{n\geq 1}^{N} a_n + \sum_{n\geq 1}^{N-1} a_n = a_N.$$

And $\lim_{N\to\infty} S_N - S_{N-1} = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = 0.$ $\square$

Divergence Test

So if $(a_n) \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example: $(a_n) \to 0$ but $\sum_{n=0}^{\infty} a_n$ diverges.

Take $\sum_{n=0}^{\infty} \frac{1}{n}$ which diverge. But $(\frac{1}{n}) \to 0$.