Discussion 9

Oct 22, 2024

Last Time:

- Summing Series
- Integral Test
- Direct Comparison

Today:

- Limit Comparison
- Alternating Series

Summary So far

- Known series:
  - Geometric Series: $\sum_{n=0}^{\infty} x^n$
  - Telescopic Series: $\sum_{n=0}^{\infty} b_n - b_{n+1}$
  - P-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

- Tools
  - test for divergence
  - Direct Comparison Test
  - Algebra with series.
  - Limit Comparison (New)
  - Alternating Series (New)
  - Absolute & Conditional Convergence.

Thm (Limit Comparison)

Let (a_n), (b_n) sequences s.t. $a_n, b_n > 0$.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ and $C>0$

then $\sum a_n$ converge iff $\sum b_n$ converge.

* If $C=\infty$ or $C=0$, limit test does not work!

- Moreover, if
  1) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ conv, then $\sum a_n$ conv

  2) $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div, then $\sum a_n$ div

Remark: The hard part is finding the right series to compare to. Below we have examples to show how "estimate behavior" helps.

- Given a sequence (b_n), a series of the form $\sum_{n=0}^{\infty} (-1)^n b_n$ is called an alternating series.

Thm (Alternating Series)

Let (b_n) sequence. If

1) $b_n \geq 0$ for all n
2) (b_n) decreasing (non-increasing)
3) $\lim_{n\to\infty} b_n = 0$, then

$$\sum_{n=0}^{\infty} (-1)^n b_n \text{ converge.}$$

Example: $(b_n) = \frac{1}{n}$.

1) $\frac{1}{n} \geq 0$ all n
2) $(\frac{1}{n})$ decreasing
3) $\lim_{n\to\infty} \frac{1}{n} = 0$

By alternating series test $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converge!

def (Absolute & Conditional Convergence)

Let (a_n) sequence.

We say $\sum a_n$ converges absolutely to mean

$\sum_{n=1} |a_n| < +\infty$.

We say $\sum a_n$ converge conditionally to mean

$\sum_{n=1}^{\infty} a_n$ converge, but $\sum_{n=1}^{\infty} |a_n|$ not convergent.

Thm (Absolute Convergence)

If $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\sum_{n=1}^{\infty} a_n$ converge

Thm (Alternating Series Estimation)

Let $b_n \geq 0$ for all n, (b_n) decreasing, and $\lim_{n\to\infty} b_n = 0$.

Let $S = \sum_{n=0}^{\infty} (-1)^n b_n$, and $S_N = \sum_{n=0}^{N} (-1)^n b_n$.

Then $|S - S_N| \leq b_{N+1}$.

Examples :

① $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+7n-1}{5n^5+n^4-3n^3+2}$ (limit comp)

② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2+1}}$ (limit comp)

③ $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{4n^3-1}$ (limit comp)

④ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+2}{3n+1}\right)^n$ (limit cop)

⑤ which of the following conv abs ?

⑥ $\displaystyle\sum_{n=1}^{\infty} \frac{5n\sqrt{n+5}}{n^5+5n^2+1}$ (limi cop)

⑦ $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\ln(n)}$ (limit comp)

Example 1:

$$\sum_{n=1}^{\infty} \underbrace{\frac{2n^2+7n-1}{5n^5+n^4-3n^3+2}}_{``a_n"}$$

• So $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2+7n-1}{5n^5+n^4-3n^3+2} = 0.$

Find (b_n) s.t ① $\sum b_n$ is easy to understand,

and ② $\lim_{n\to\infty} a_n \cdot \frac{1}{b_n} =$ constant >0, and finite.

Well $\lim_{n\to\infty} \frac{2n^2+7n-1}{5n^5+n^4-3n^3+2} \cdot n^3 = \frac{2}{5} > 0$

So $b_n = \left(\frac{1}{n^3}\right)$? Yes $\sum \frac{1}{n^3}$ converge by $p=3$ test.

Hence $\sum a_n$ converge by limit comparison.

Example 2:

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{4n^3-1}$$

- $0 \le \sin^2(n) \le 1$, so $\frac{\sin^2(n)}{4n^3-1}$ "behaves" like $\left(\frac{1}{n^2}\right)$

$$\left| \frac{\sin^2(n) \cdot n^2}{4n^3-1} \right| \le \frac{n^2}{4n^3-1} \longrightarrow 0.$$

By Generalized limit comparison, since $\sum \frac{1}{n^2}$ conv, $\sum \frac{\sin^2(n)}{4n^3-1}$ conv.

More Examples :

Example 3

$$\sum_{n=1}^{\infty} \left(\frac{n+2}{3n+1}\right)^n$$

Behave like $\left(\frac{1}{3}\right)^n$, try

Example 4 :

$$\sum_{n=1}^{\infty} \frac{5n\sqrt{n+5}}{n^5+5n^2+1}$$

Example 5 :

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{\ln(n)}$$