Discussion Notes 1

Aug 27, 2024

Introduction and L1

Today

- Introductions
- L1 : Integration by parts

Example (Standard)

$$\int x \sec(x)^2 dx$$

Idea : $\frac{d}{dx}\tan(x) = \sec^2(x)$

$u = x$, $v = \tan(x)$

$du = dx$, $dv = \sec^2(x)dx$

So $\int x\sec(x)^2 dx = x \cdot \tan(x) - \int \tan(x)\,dx$

$= x \cdot \tan(x) + \int \frac{-\sin(x)}{\cos(x)}\,dx$

$$\int x \sec(x)^2 = x\tan(x) + \ln|\cos(x)| + C$$

Example 2 (Introduce 1)

$$\int_1^e (\ln x)^2\, dx \qquad \text{(solution: } e-2)$$

- Make $u = \ln(x)^2$, $v = x$
  $du = 2\ln(x) \cdot \frac{1}{x} dx$, $dv = 1\, dx$

Because the $\frac{1}{x}$ is going to cancel the v's every time.

- $\int \ln(x)^2 dx = \ln(x)^2 \cdot x - \int x \cdot 2\ln(x) \cdot \frac{1}{x}\, dx$

$= \ln(x)^2 \cdot x - 2\int \ln(x)\, dx$

Again $u = \ln(x)$, $v = x$
$du = \frac{1}{x} dx$, $dv = 1\, dx$

$\ln(e) = 1$
$\ln(1) = 0$

- So $\ln(x)^2 \cdot x - 2\int \ln(x)\, dx$

$= \ln(x)^2 \cdot x - 2\left(\ln(x) \cdot x - \int x \cdot \frac{1}{x}\, dx \right)$

$= \ln(x)^2 \cdot x - 2\ln(x) \cdot x + 2x + c$

Then $\int_1^e \ln(x)^2 dx = \left(\ln(x)^2 \cdot x - 2\ln(x) \cdot x + 2x \right) \Big|_1^e$

$= (1 \cdot e - 2 \cdot 1 \cdot e + 2 \cdot e) - (0 + 0 + 2)$

$= \boxed{e + 2}$

Question: Which of the following can be solved by u-substitution or IBP?

1.) $\int \cos x e^{\sin x}\, dx$

2.) $\int e^{2t} \cos(4t)\, dt$

Example 2 ("Cycling")

1. $\int e^{2t}\cos(4t)\,dt$ (solution: $\dfrac{e^{2t}\sin(4t)}{5} + \dfrac{e^{2t}\cos(4t)}{10} + C$)

- Both e^{2t} and $\cos(4t)$ "cycles" back and forth while taking derivatives.
  - $\frac{d}{dt}\cos(4t) = -4\sin(4t)$, $\frac{d}{dt} -4\sin(4t) = -16\cos(4t)$, ... Back to cos.
  - $\frac{d}{dt} e^{2t} = 2e^{2t}$, $\frac{d}{dt} 2e^{2t} = 4e^{2t}$, ...

Introduce $I = \int e^{2t}\cos(4t)\,dt$ (call this quantity I.)

- Choose $u = \cos(4t)$, $v = \frac{1}{2}e^{2t}$
  $du = -4\sin(4t)\,dt$, $dv = e^{2t}\,dt$

Then $I = \cos(4t)\cdot\frac{1}{2}e^{2t} - \int \frac{1}{2}e^{2t}\cdot(-4\sin 4t)\,dt$
$= \cos(4t)\frac{e^{2t}}{2} + 2\int e^{2t}\sin(4t)\,dt.$

- Chose $u = \sin(4t)$, $v = \frac{1}{2}e^{2t}$
  $du = 4\cos(4t)\,dt$, $dv = e^{2t}\,dt.$

Then $I = \cos(4t)\frac{e^{2t}}{2} + 2\left(\sin(4t)\cdot\frac{e^{2t}}{2} - \int \frac{1}{2}e^{2t}\cdot 4\cos(4t)\,dt\right)$

$\Rightarrow I = \cos(4t)\frac{e^{2t}}{2} + 2\sin(4t)\cdot e^{2t} - 4\overbrace{\int e^{2t}\cdot\cos(4t)\,dt}^{I}$

$\Rightarrow I^{+4I} = \cos(4t)\frac{e^{2t}}{2} + 2\sin(4t)\cdot e^{2t} - 4I^{+4I}$

$\Rightarrow 5I = \cos(4t)\frac{e^{2t}}{2} + 2\sin(4t)\cdot e^{2t}$

$\Rightarrow \boxed{I = \frac{1}{5}\left(\cos(4t)\frac{e^{2t}}{2} + \sin(4t)\cdot e^{2t}\right) + C}$

Example 3 (Repeated IBP)

2. $\int_0^1 x^3 e^{2x}\,dx$ (solution: $\frac{e^2+3}{8}$)

Apply IBP three times and chose "u's" as x^3, x^2, x.

Example (Introduce 1)

7. $\int_0^{1/2} \arccos x \, dx$ (solution: $\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$)

$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}$

Introduce $dv = 1\,dx$, $v = x$

Choose $u = \arccos(x)$, $du = \frac{-1}{\sqrt{1-x^2}}$

Then $\int \arccos(x)\,dx = \arccos(x) \cdot x - \int \frac{-x}{\sqrt{1-x^2}}\,dx$

For $\int \frac{-x}{\sqrt{1-x^2}}\,dx$ choose $w = 1-x^2$, $dw = -2x\,dx$

$\Rightarrow \frac{1}{2}dw = -x\,dx$

So $\int \frac{-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{2\sqrt{w}}\,dw$

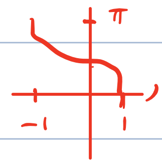
$= \int \frac{1}{2} w^{-\frac{1}{2}}\,dw$

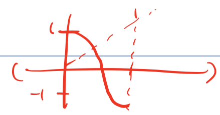
$= w^{\frac{1}{2}} + c$

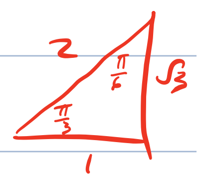
$= (1-x^2)^{\frac{1}{2}} + c$

$\arccos(\frac{1}{2}) = \frac{\pi}{3}$

$\arccos(0) = \frac{\pi}{2}$







Then $\int_0^{\frac{1}{2}} \arccos(x)\,dx = \left(x \cdot \arccos(x) - \sqrt{1-x^2}\right)\Big|_0^{\frac{1}{2}}$

$= \left(\frac{1}{2} \cdot \frac{\pi}{3} - \sqrt{\frac{3}{4}}\right) - (0 - 1)$

$$\boxed{\int_0^{\frac{1}{2}} \arccos(x)\,dx = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1}$$