Discussion Notes 2

Sept 3, 2024

Last Time

L1: Integration by parts

Today:

L2: More integration by parts

L3: Trigonometric integrals 1

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

Remember

$$\cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$$

And one can calculate:

$$\cos^2(x) = \sin^2(x) + \underbrace{\cos^2(x) - \sin^2(x)}$$

$$\Rightarrow \cos^2(x) = \sin^2(x) + \cos(2x)$$

$$\Rightarrow \cos^2(x) = 1 - \cos^2(x) + \cos(2x)$$

$$\Rightarrow 2\cos^2(x) = 1 + \cos(2x)$$

$$\Rightarrow \cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$$

Problem 1

$$\int \tan^3(x) \cos^4(x)\, dx$$

$$= \int \frac{\sin^3(x) \cdot \cos^4(x)}{\cos^3(x)}\, dx$$

$$= \int \sin^3(x) \cdot \cos(x)\, dx$$

$$= \sin^4(x)/4 + C$$

Problem 2:

$$\int \cos^{2020}(x) \cdot \sin^3(x)\, dx$$

$$= \int \cos^{2020}(x) \cdot \sin(x) \cdot (1 - \cos^2(x))\, dx$$

$$= \int \cos^{2020}(x) \sin(x)\, dx - \int \cos^{2022}(x) \cdot \sin(x)\, dx$$

$$= -\frac{\cos^{2021}(x)}{2021} + \frac{\cos^{2023}(x)}{2023} + C$$

Problem 3 :

6. $\displaystyle\int \frac{\cos^5 x}{\sqrt{\sin x}}\,dx$ (solution: $2\sqrt{\sin x} - \frac{4}{5}(\sin x)^{5/2} + \frac{2}{9}(\sin x)^{9/2} + C$)

$$\int \cos^5(x) \cdot \sin(x)^{-\frac{1}{2}}\,dx$$

$$= \int \cos^3(x) \cdot (1 - \sin^2(x)) \cdot \sin^{-\frac{1}{2}}(x)\,dx$$

$$= \int \cos^3(x) \cdot \sin^{-\frac{1}{2}}(x)\,dx - \cos^3(x) \cdot \sin^{3/2}(x)\,dx$$

$$= \int \cos(x)\,(1 - \sin^2(x))\,\sin^{-\frac{1}{2}}(x)\,dx - \int \cos^3(x) \cdot \sin^{3/2}(x)\,dx$$

$$= \int \cos(x) \cdot \sin^{-\frac{1}{2}}(x) - \cos(x) \cdot \sin^{3/2}(x)\,dx - \int \cos^3(x) \cdot \sin^{3/2}(x)\,dx$$

$$= \int \cos(x) \cdot \sin^{-\frac{1}{2}}(x) - \cos(x) \cdot \sin^{3/2}(x)\,dx - \int \cos(x)\,(1 - \sin^2(x)) \cdot \sin^{3/2}(x)\,dx$$

$$= \int \cos(x) \cdot \sin^{-\frac{1}{2}}(x) - \int \cos(x) \cdot \sin^{3/2}(x)\,dx - \int \cos(x) \cdot \sin^{3/2}(x) - \cos(x) \cdot \sin^{7/2}(x)\,dx$$

$$= \frac{\sin(x)^{\frac{1}{2}}}{1/2} - \frac{\sin(x)^{5/2}}{5/2} - \frac{\sin(x)^{5/2}}{5/2} + \frac{\sin(x)^{9/2}}{9/2} + C$$

$$= \boxed{2\sin(x)^{1/2} - \frac{4}{5}\sin(x)^{5/2} + \frac{2}{9}\sin(x)^{9/2} + C}$$

A shorter way ; $u = \sin(x)$, $du = \cos(x)\,dx$

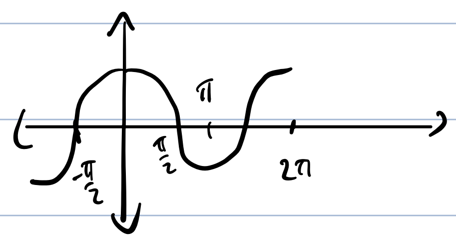
$$\int \frac{\cos^4(x)}{\sqrt{u}}\,du = \int \frac{((1 - \sin^2(x))^2}{\sqrt{u}}\,du = \int \frac{(1 - u^2)^2}{\sqrt{u}}\,du.$$

Problem 4:

7. Explain why the following calculation is incorrect, and then find the correct value of the integral. (solution: $2+\sqrt{3}/2$)

$$\begin{aligned}\int_{-\pi/3}^{\pi}\sqrt{1-\sin^2 x}\,dx &= \int_{-\pi/3}^{\pi}\sqrt{\cos^2 x}\,dx \\ &= \int_{-\pi/3}^{\pi}\cos x\,dx \\ &= [\sin x]_{-\pi/3}^{\pi} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\int_{-\pi/3}^{\pi}\sqrt{\cos^2(x)}\,dx = \int_{-\pi/3}^{\pi/2}\cos(x)\,dx + \int_{\pi/2}^{\pi}-\cos(x)\,dx$$



$$= \sin(x)\Big|_{-\pi/3}^{\pi/2} - \left(\sin(x)\Big|_{\pi/2}^{\pi}\right)$$

$$= \left(1-\left(-\frac{\sqrt{3}}{2}\right)\right) - (0-1)$$

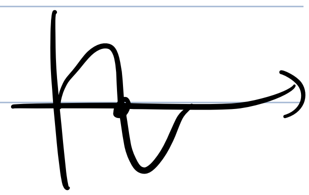
$\sin(\frac{\pi}{2})=1$

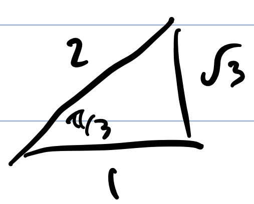
$$= 1+\frac{\sqrt{3}}{2}+1$$

$\frac{\sqrt{3}}{2}$

$\sin(-\frac{\pi}{3})$

$$= 2+\sqrt{3}/2$$





$$\boxed{\int_{-\pi/3}^{\pi}\sqrt{\cos^2(x)}\,dx = 2+\sqrt{3}/2}$$

Problem (Sp. 2024, Exam 1, nr. 5)

$$\int_0^{\pi/2} \sin^2(x)\cos^3(x)\,dx = \int_0^{\frac{\pi}{2}} \sin^2(x)\cdot(1-\sin^2(x))\cdot\cos(x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\,dx - \int_0^{\frac{\pi}{2}} \sin^4(x)\cdot\cos(x)\,dx$$

$$= \frac{\sin^3(x)}{3}\Big|_0^{\frac{\pi}{2}} - \frac{\sin^5(x)}{5}\Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{3}-0\right) - \left(\frac{1}{5}-0\right)$$

$$= 5/15 - 3/15 = 2/15$$

E

Problem

4. $\displaystyle\int x\cos^2 x\,dx$

$$\int x\cos^2(x)\,dx = \int x\left(\frac{1}{2}+\frac{\cos(2x)}{2}\right)dx$$

$$= \int \frac{x}{2}\,dx + \frac{1}{2}\int x\cdot\cos(2x)\,dx$$

Then $u = x$, $v = \frac{1}{2}\sin(2x)$

$du = dx$, $dv = \cos(2x)\,dx$

$$\int x\cos(2x)\,dx = x\cdot\frac{1}{2}\sin(2x) - \int\frac{1}{2}\sin(2x)\,dx$$

$$= \frac{x}{2}\sin(2x) + \frac{1}{2}\cdot\frac{1}{2}\cos(2x)$$

$$= \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)$$

So $\displaystyle\int x\cos^2(x)\,dx = \frac{x^2}{4} + \frac{1}{2}\left(x\cdot\frac{1}{2}\sin(2x) + \frac{1}{4}\cos(2x)\right) + C$

$$\boxed{\int x\cos^2(x)\,dx = \frac{x^2}{4} + \frac{x}{4}\sin(2x) + \frac{1}{8}\cos(2x) + C}$$

Problem: Evaluate $\int \sec(x)\,dx$?

[Trick is to multiply $\sec(x) = \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)$]

Since $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

$\frac{d}{dx}(\tan(x)) = \sec^2(x)$

$$\int \sec(x)\,dx = \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

Choose $u = \sec(x) + \tan(x)$, then $du = (\sec^2(x) + \sec(x)\tan(x))\,dx$

So we get $\int \frac{1}{u}\,du = \ln|u| + C$.

Hence $\boxed{\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C.}$

Problem : Evaluate $\int \cos^4(x)\,dx$?

[trick is to use identity $\cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$.]