# Discussion Notes 4 — Sept , 2024

## Last Time

- Trigonometric Substitution

## Today

- Partial Fraction Decomposition.

| Case | Factor in denominator | Term in PFD |
|---|---|---|
| 1 | $ax+b$ | $\dfrac{A}{ax+b}$ |
| 2 | $(ax+b)^n$ | $\dfrac{A_1}{ax+b}+\cdots+\dfrac{A_n}{(ax+b)^n}$ |
| 3 | $ax^2+bx+c$ | $\dfrac{Ax+B}{ax^2+bx+c}$ |
| 4 | $(ax^2+bx+c)^n$ | $\dfrac{A_1x+B_1}{ax^2+bx+c}+\cdots+\dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$ |

**Extra Problems:**

dn $\displaystyle\int \frac{x-3}{x^2-4x-5}\,dx$ (solution: $\frac{1}{3}\ln|x-5|+\frac{2}{3}\ln|x+1|+C$)

2. $\displaystyle\int_0^6 \frac{3x-20}{x^2-14x+49}\,dx$ (solution: $\frac{6}{7}-3\ln 7$)

3. $\displaystyle\int \frac{x^3-9x+15}{x^2+x-6}\,dx$ (solution: $\frac{x^2}{2}-x+\ln|x-2|-3\ln|x+3|+C$)

4. $\displaystyle\int \frac{ds}{s^2(s-1)^2}$ (solution: $2\ln|s|-\frac{1}{s}-2\ln|s-1|-\frac{1}{s-1}+C$)

fw $\displaystyle\int \frac{e^{2x}}{(e^x+3)^2(e^x-2)}\,dx$ (solution: $\frac{-2\ln(e^x+3)}{25}-\frac{3}{5(e^x+3)}+\frac{2\ln|e^x-2|}{25}+C$)

6. $\displaystyle\int \frac{x^2-5x-9}{(x-1)^3}\,dx$ (solution: $\ln|x-1|+\frac{3}{x-1}+\frac{13}{2(x-1)^2}+C$)

7. $\displaystyle\int_0^1 \frac{2}{x^2+4x+3}\,dx$ (solution: $\ln 3/2$)

8. $\displaystyle\int_0^1 \frac{2}{x^2+4x+4}\,dx$ (solution: $1/3$)

**MAC 2312 L8 - 5**

**Extra Problems:**

1. $\displaystyle\int \frac{x}{(x^2+4x+13)^2}\,dx$ (solution: $-\frac{2x+13}{18(x^2+4x+13)}-\frac{1}{27}\arctan\left(\frac{x+2}{3}\right)+C$)

2. $\displaystyle\int \frac{x^3+2x^2+2x+2}{(x^2+1)(x^2+2)}\,dx$ (solution: $\frac{1}{2}\ln(x^2+1)+\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)+C$)

3. $\cancel{\displaystyle\int \frac{x^3+4}{x^2+4}\,dx}$ (solution: $\frac{x^2}{2}-2\ln(x^2+4)+2\arctan\left(\frac{x}{2}\right)+C$)

4. $\displaystyle\int_{-3}^1 \frac{2x+3}{x^2+6x+25}\,dx$ (solution: $\ln 2-\frac{3\pi}{16}$)

5. $\displaystyle\int \frac{x^2+2x+7}{(x^2+4)^2}\,dx$ (solution: $\frac{11}{16}\arctan\left(\frac{x}{2}\right)+\frac{3x-8}{8(x^2+4)}+C$)

6. $\displaystyle\int_1^4 \frac{1}{x^2+x\sqrt{x}}\,dx$ (Hint: Let $u=\sqrt{x}$) (solution: $1+\ln\frac{9}{16}$)

# So Far

- U-substitution

  $u = f(x)$, $du = f'(x)dx$

  Need $f'(x)$ inside integral

- Integration by Parts

  $\int u\,dv = uv - \int v\,du$

  Hopefully $\int v\,du$ easier than $\int u\,dv$

- Trigonometric Integrals

  $\int \sec(x)\,dx = \ln|\sec x + \tan(x)|$

  $\vdots$

  Apply identities, and u-sub, IBP

- Trigonometric substitution

  $x = a\sin\theta$    $a^2 - x^2$

  $x = a\sec\theta$    $x^2 - a^2$

  $x = a\tan\theta$    $x^2 + a^2$

  Helps with cancelling $\sqrt{\cdot}$

- Partial Fraction Decomposition
  - $\frac{1}{x}$, $\frac{1}{x^2}$, ...
  - $\frac{1}{x^2+1}$, $\frac{1}{(x^2+1)^2}$, ...

  $\frac{p}{q}$, $p, q$ polynomials, $\deg(p) < \deg(q)$.

**Problem 2**:

1. $\int \frac{x-3}{x^2-4x-5}\,dx$ (solution: $\frac{1}{3}\ln|x-5| + \frac{2}{3}\ln|x+1| + C$)

$(x-5)(x+1) = x^2 - 4x - 5.$

$$\frac{x-3}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1} = \frac{A(x+1)+B(x-5)}{(x-5)(x+1)}.$$

Then $Ax + A + Bx - 5B = (A+B)x + A - 5B.$

$\Rightarrow \quad A+B=1$

$- (A-5B=-3)$

$6B = 4 \quad \Rightarrow \quad B = \frac{4}{6} = \frac{2}{3}$

$A = 1-\frac{2}{3} = \frac{1}{3} \quad \Rightarrow \quad A = \frac{1}{3}$

So $\frac{1}{3}\int \frac{1}{x-5}\,dx + \frac{2}{3}\int \frac{1}{x+1}\,dx$

$= \boxed{\frac{1}{3}\ln|x-5| + \frac{2}{3}\ln|x+1| + C.}$

# Problem 5 :

$$e^x \cdot e^x$$

5. $\int \frac{e^{2x}}{(e^x+3)^2(e^x-2)}\,dx$ (solution: $\frac{-2\ln(e^x+3)}{25} - \frac{3}{5(e^x+3)} + \frac{2\ln|e^x-2|}{25} + C$)

A C

$$\frac{(e^x)^2}{(e^x+3)^2(e^x-2)}$$

$$u = e^x$$

$$du = e^x\,dx$$

$$\int \frac{u}{(u+3)^2(u-2)}\,du$$

$$\frac{u}{(u+3)^2(u-2)} = \frac{A}{(u+3)} + \frac{B}{(u+3)^2} + \frac{C}{u-2}$$

$$= \frac{A(u+3)(u-2) + B(u-2) + C(u+3)^2}{(u+3)^2(u-2)}$$

Then $A(\cancel{u^2} \cancel{+u} \cancel{-6}) + \cancel{Bu} - \cancel{2B} + C(\cancel{u^2} + \cancel{6u} \cancel{+9})$

$$= Au^2 + Cu^2 + Au + Bu + 6Cu - 6A - 2B + 9C$$

$$= (A+C)u^2 + (A+B+6C)u + (-6A-2B+9C)$$

So
$$A + C = 0 \Rightarrow A = -C$$
$$A + B + 6C = 1$$
$$-6A - 2B + 9C = 0$$

$$2A + 2B + 12C = 2$$
$$+(-6A - 2B + 9C) = 0$$
$$-4A + 0 + 21C = 2$$
$$-4(-C) + 21C = 2$$
$$25C = 2$$
$$C = \tfrac{2}{25}, \quad A = \tfrac{-2}{25}$$

And $\frac{-2}{25} + B + \frac{12}{25} = \frac{25}{25}$

$\Rightarrow \quad B + \frac{10}{25} = \frac{25}{25}$

$\Rightarrow \quad B = \frac{15}{25} = \frac{3}{5}$

$$\frac{u}{(u+3)^2(u-2)} = \frac{-2}{25}\frac{1}{(u+3)} + \frac{3}{25}\cdot\frac{1}{(u+3)^2} + \frac{2}{25}\cdot\frac{1}{u-2}$$

$$\int \frac{u}{(u+3)^2(u-2)}\,du = \frac{-2}{25}\ln|u+3| - \frac{3}{5}\cdot\frac{1}{(u+3)} + \frac{2}{25}\ln|u-2|$$

$$= \frac{-2}{25}\ln|e^x+3| - \frac{3}{5}\cdot\frac{1}{(e^x+3)} + \frac{2}{25}\ln|e^x-2|$$

$(u+3)^{-2} =$

**Problem**

3. $\int \frac{x^3+4}{x^2+4}\,dx$ (solution: $\frac{x^2}{2} - 2\ln(x^2+4) + 2\arctan\left(\frac{x}{2}\right) + C$ )

- numerator > denominator

$$\begin{array}{r|l} & x \\ \hline x^2+4 & x^3+4 \\ & x^3+4x \\ \hline & -4x+4 \end{array}$$

So $\frac{x^3+4}{x^2+4} = x + \frac{-4x+4}{x^2+4} = x - \frac{4x}{x^2+4} + \frac{4}{x^2+4}$

$\int x\,dx = x^2/2 + C$

$\int \frac{4x}{x^2+4}\,dx = \int \frac{2}{u}\,du = \ln|u|$ , $u = x^2+4$

$du = 2x\,dx$

$= 2\ln|x^2+4| + C$ $\quad 2du = 4x\,dx$

$\int \frac{4}{x^2+4}\,dx = 4\int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta+4} = 2\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta}$

$x = 2\tan\theta$ $\quad = 2\int d\theta$

$x = 2\sec^2\theta\,d\theta$ $\quad = 2\arctan\left(\frac{x}{2}\right) + C$

**Example**

Extra Problems:

1. $\displaystyle\int \frac{x}{(x^2+4x+13)^2}\,dx$ (solution: $-\dfrac{2x+13}{18(x^2+4x+13)} - \dfrac{1}{27}\arctan\left(\dfrac{x+2}{3}\right) + C$)