Homework 10

Due April 2, 2025

Solutions and Rubric

§4.3 : 1, 2, 9, 10, 12, 15, 17, 20, 21

§8.1 : 1, 4(b), 5(a), 9(a), 15(a), 21(b)

Question 1 (§ 4.3, nr. 1)

a.) False, $E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ elementary matrix with $\det(E) = 2$.

b.) True, see thm 4.7

c.) False, $A \in M_n(\mathbb{C})$ we have A^{-1} exists iff $\det(A) \neq 0$.

d.) True, since for $A \in M_n(\mathbb{C})$ full rank is equivalent to invertable.

e.) False, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\det(A) = \det(A^t)$

F) True

g.) False, only when coefficient matrix is invertable.

h.) False, M_k obtained by replacing k^{th} column by b.

Question 2 (§ 4.3, nr. 2)

Use Cramer's Rule to solve the following system's of equations:

2.)
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2 \quad (*)$$

Where $a_{11}a_{22} - a_{21}a_{12} \neq 0$

$(*)$ Can be rewritten $Ax=b$ where
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then $M_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$, $M_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$

And
$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

3.
$$2x_1 + x_2 - 3x_3 = 5$$
$$x_1 - 2x_2 + x_3 = 10$$
$$3x_1 + 6x_2 - 2x_3 = 0$$

This can be rewritten $Ax=b$ where

$$\begin{bmatrix} 2 & 1 & -3 \\ 0 & -2 & 1 \\ 3 & 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 5 & 1 & -3 \\ 10 & -2 & 1 \\ 0 & 6 & -2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 2 & 1 & 5 \\ 0 & -2 & 10 \\ 3 & 6 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 2 & 5 & -3 \\ 0 & 10 & 1 \\ 3 & 0 & -2 \end{bmatrix}$$

By Cramer's Rule:

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{-100}{-15} = \frac{20}{3}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{65}{-15} = -\frac{13}{3}$$

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{-20}{-15} = \frac{4}{3}$$

Question 3 (§ 4.3, nr. 9)

Show that an upper-triangular $n \times n$ matrix is invertable iff all its main-diagonal entries are non-zero.

Pf: By exercise 23, §4.2 we showed

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$, where A_{ii} the (i,i) entry of A.

Also as a consequence of the determinant being multiplicative, we know $\det(A) \neq 0$ iff A is invertable.

That is $\prod_{i=1}^{n} a_{ii} \neq 0$ iff A is invertable.

Equivalently, $a_{ii} \neq 0$ $\forall i$ iff A is invertable.

$\blacksquare$

Question 4 (§ 4.3, nr. 10)

A matrix $A \in M_n(F)$ is called nilpotent to mean there exists a positive integer k s.t $A^k = 0$.

Show that nilpotent matrices have 0 det.

Pf: Indeed using the fact that determinants are multiplicative we get.

Let $k \in \mathbb{N}$ s.t $A^k = 0$. Then

$$
\begin{aligned}
0 = \det(A^k) &= \det(A)\det(A^{k-1}) \\
&= \det(A) \cdots \det(A) \\
&= \det(A)^k .
\end{aligned}
$$

$\Rightarrow \det(A) = 0$

$\blacksquare$

Remark:

We also showed that nilpotent matrices cannot be invertable.

Question 5 (§4.3, nr. 12)

Given $Q \in M_n(\mathbb{R})$ we say Q is <u>orthogonal</u> to mean $QQ^T = I$.

(These matrices are very important in numerical analysis)

Show that if Q is orthogonal, then $\det(Q) = \pm 1$.

Pf: Indeed since recall $x^2 = 1 \Rightarrow x = \pm 1$. So

$$\begin{aligned} \det(Q)^2 &= \det(Q)\det(Q) \\ &= \det(Q)\det(Q^T) \\ &= \det(QQ^T) \\ &= \det(I) \\ &= 1 \end{aligned}$$

det. invariant under transp.

$\Rightarrow \det(Q) = 1$ or $\det(Q) = -1$

Question (§4.3, nr. 13)

This next results says that the determinants of unitaris lie on the circle.

Given $U \in M_n(\mathbb{C})$. Let $\overline{U}$ denote the matrix where we apply complex conjugation to each entry. By U^* we mean $(\overline{U})^T$, the conjugate transpose of U.

We say U is <u>unitary</u> to mean $UU^* = I$. That is all unitaries are invertable, and $U^{-1} = U^*$.

a) Show that $\det(\overline{U}) = \overline{\det(U)}$.

b) If U is unitary, then $|\det(U)| = 1$

For a), $n=2$ follows by direct computation:

Assume $\det(\overline{U}) = \overline{\det(U)}$ matrices of size n.

Then $$\det(\overline{U}) = \sum_{j=1}^{n+1} (-1)^{1+j} \cdot \overline{U_{1j}} \cdot \det(\overline{\tilde{U}_{1j}})$$

$\downarrow$ induction hypothesis.

$$= \sum_{j=1}^{n+1} (-1)^{1+j} \cdot \overline{U_{1j}} \cdot \overline{\det(\tilde{U}_{1j})}$$

$$\sum_{j=1} (-1)^{1+j} \cdot u_{1,j} \cdot \det(u_{1,j})$$

$$= \overline{\sum_{j=1}^{n+1} (-1)^{1+j} \cdot u_{1,j} \cdot \det(\tilde{u}_{1,j})}$$

$$= \overline{\det(u)} .$$

For b.). Recall $|z|^2 = z \cdot \bar{z}$ for $z \in \mathbb{C}$.

Observe that

$$|\det(u)|^2 = \det(u) \cdot \overline{\det(u)}$$

$$= \det(u) \cdot \det(\bar{u}) \quad \text{part a)}$$

$$= \det(u) \cdot \det((\bar{u})^t) \quad \text{det pres under } t.$$

$$= \det(u) \cdot \det(u^*) \quad u^{-1} = u^*$$

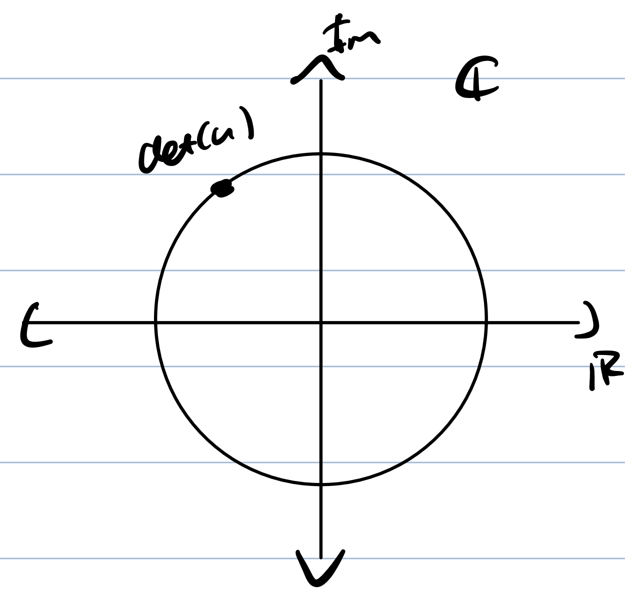
$$= \det(u) \cdot \det(u^{-1})$$

$$= \det(uu^{-1})$$

$$= \det(I)$$

$$= 1 .$$

So $|\det(u)| = 1.$



Remark: Recall 2×2 interpretation of $|\det(u)|$ as the area of a parallelogram.

What does $|\det(u)| = 1$ mean for the shape of parallelogram formed by rows of u?

Question 6 (§4.3, nr. 15)

This is needed to define char. poly of linear maps $T: V \to V$.

Show that the determinant does not depend on your choice of basis.

That is show that similar matrices have the same determinant.

Pf: Suppose $A, B \in M_n(F)$ and similar.
That is $\exists$ invertible $S \in M_n(F)$ s.t

$$A = SBS^{-1}.$$

Then
$$\begin{aligned}
\det(A) &= \det(SBS^{-1}) \\
&= \det(S)\det(B)\det(S^{-1}) && \text{Thm 4.7} \\
&= \det(S)\cdot\det(S)^{-1}\det(B) && \text{Cor pg 223} \\
&= \det(B)
\end{aligned}$$

as required. $\blacksquare$

Question 7 (§4.3, nr. 17)

Recall that we say a field F has char. $p \in \mathbb{N}$ to $\sum_{i=1}^{p} 1 = 0$, and p is the smallest such integer. If no such p exists, we say F has char. 0.

Show that if $AB = -BA$, $A, B \in M_n(F)$, n odd, F not characteristic 2, then A or B is not invertable.

Pf: We know
$$\det(AB) = \det(-BA)$$
$$= (-1)^n \det(BA)$$
$$= -\det(BA)$$

$\Rightarrow \det(AB) + \det(BA) = 0$

$\Rightarrow \det(A)\det(B) + \det(B)\det(A) = 0$

$\Rightarrow 2\det(A)\det(B) = 0$

$\Rightarrow$ at least one of $\det(A) = 0$ or $\det(B) = 0$.

That means at least one of A or B not invertable. $\boxed{\Xi}$

Question 8 (§4.3, nr. 20.)

Suppose $M \in M_n(F)$ and can be written in block triangular form

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}.$$

where A is square. Show that $\det(M) = \det(A)$.

Pf. One way to show this is to do co-factor expansion along the bottom row n.

$$M = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1k} & & B & \\ a_{k1} & \cdots & a_{kk} & & & \\ \hline & & & 1 & & \\ & O & & & 1 & \\ & & & & & \ddots \; 1 \end{array}\right)$$

Step 1: Set $M^{(1)} = M$

$$\det(M^{(1)}) = \sum_{j=1}^{n} (-1)^{n+j} \cdot M_{nj} \cdot \det(\tilde{M}_{nj})$$

$$= (-1)^{n+n} \cdot M_{nn} \cdot \det(\tilde{M}_{nn}) \quad , \; M_{nn} = 1$$

$$= \det(\tilde{M}_{nn})$$

Step 2: Set $M^{(2)} = \tilde{M}_{nn}$

Then $\det(M^{(2)}) = \sum_{j=1}^{n-1} (-1)^{n-1+j} \cdot M^{(2)}_{n-1,j} \cdot \det(M^{(2)}_{n-1,j})$

$m_{n-1,n-1} = 1$

$= \det(M^{(2)}_{n-1,n-1})$

Continue this process for ℓ-steps where ℓ is the size of I in lower right block.

Step ℓ: We end with $\det(M) = \det(A)$

$\circledcirc$

Question 9 (§ 4.3, nr 21)

Prove that if $M \in M_{n\times n}(F)$ can be written in block triangular form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$ where A, C square matrices, of size k, n

then $\det(M) = \det(A) \cdot \det(C)$

Proof: See below on why M is invertable iff main diagonal blocks are invertable.

If C is not invertable, M is not

$\Rightarrow \det(M) = 0 = \det(A)\det(C)$.

If C is invertable, then

$$\begin{pmatrix} \underline{\underline{I}} & 0 \\ 0 & C^{-1} \end{pmatrix} \overset{M}{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}} = \begin{pmatrix} A & B \\ 0 & \underline{\underline{I}} \end{pmatrix}$$

$\Rightarrow \det(C^{-1}) \cdot \det(M) = \det(A)$

$\Rightarrow \det(M) = \underline{\det(A)}$

$\det(C^{-1})$

$= \det(A) \cdot \det(C)$

$\boxed{\Xi}$

Pf: (long vesion)

First show that M is invertable iff both main diagonal blocks are inverstable.

Suppose M is invertable, meaning there exists

$$M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \text{ s.t.}$$

$$MM^{-1} = I \quad \text{and} \quad M^{-1}M = I$$

$$\Rightarrow \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} AE + BG & AF + BH \\ CG & CH \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_n \end{bmatrix}$$

From the above equation one needs of

- $AE + BG \quad I_k$
- $AF + BH \quad 0$
- $CG \quad 0$
- $CH \quad I_n$

Firstly $CH = I_n$, and since left inverse $\Rightarrow$ right inverse we get C^{-1} exists and equal to H.

Next since $CG = 0$, and C invertable, this means $G = 0$. Therefor $AE + 0 = I_k \Rightarrow A$ invertable, as required.

Conversely, if A and C are invertable, observe that

$$\left[\begin{array}{cc|cc} A & B & I & 0 \\ 0 & C & 0 & I \end{array}\right] \xrightarrow[C^{-1}R_2]{A^{-1}R_1} \left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & I & 0 & C^{-1} \end{array}\right]$$

$$\xrightarrow{R_1 - A^{-1}BR_2} \left[\begin{array}{cc|cc} I & 0 & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

Now this means if at least one of A or B not invertable we get M not invertable, so

$$\det(M) = 0 = \det(A)\det(B).$$

Hence we can assume both A, B invertable.

Well

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \xrightarrow{R_1 - BC^{-1}R_2} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \xrightarrow{C_2 C^{-1}} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

So we get

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & BC^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}$$

Then

$$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det\begin{pmatrix} I & BC^{-1} \\ 0 & I \end{pmatrix} \det\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \det\begin{pmatrix} I & \\ 0 & C \end{pmatrix}$$

$$= \det(I)\cdot\det(I)\det(A)\cdot\det(I)\det(I)\det(C)$$

$$= \det(A)\det(C)$$

where the second step follows from Q20, and $\det\begin{pmatrix} I & 0 \\ 0 & C\end{pmatrix} = \det(C)$ just a modification of proof of Q20. $\boxed{\Sigma}$

$$\det\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix} = t^2+1$$

Question 10 (§5.1)

a.) False, it depends over which field you are working, example $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has no eigenvalues in $\mathbb{R}$.

b.) True, if $Tv = \lambda v$, then $T(\alpha v) = \lambda(\alpha v)$ for any scalar $\alpha \in \mathbb{R}$.

c.) True, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as a square matrix over $\mathbb{R}$ has no eigen vectors, since it has no eigenvalue.

d.) False, eigenvectors is non-zero by definition.

e.) False

f.) False, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$, but 2 is not an eigenvalue.

g.) False, consider the backward shift, on the vector space of sequences $\{(a_0, a_1, ..) \mid a_i \in F\}$

$B(a_0, a_1, a_2, \ldots) := (a_1, a_2, \ldots)$

Then the sequence $(1, 1, 1, \ldots)$ is an eigenvec, with eigenvalue 1.

h.) True.

i.) True, since if $A = SBS^{-1}$, det $\det(tI - A) = \det(tI - B)$

j.) False,

k.) False, $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, But $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector since $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Question 11 (SS.1, nr 4.)

For each of the following matrices $A \in M_{n\times n}(F)$

(i) Determine all eigenvalues of A

(ii) For each eigenvalue λ of A find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n of eigen vectors of A

(iv) If succesfull in finding a basis of eigenvectors, determine an invertable matrix Q, and diagonal matrix D s.t $Q^{-1}AQ = D$.

(v) Compare dimension of eigenspace to multiplicity of root of eigenvalue.

a.) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

well $f_A(t) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - tI \right)$

$= (1-t)(2-t) - 6$

$= 2 - 3t + t^2 - 6$

$= t^2 - 3t - 4$

$= (t-4)(t+1)$

So eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 4$

- $\operatorname{Ker}(A + I) = \operatorname{Ker}\left(\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \right) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

- $\operatorname{Ker}(A - 4I) = \operatorname{Ker}\left(\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \right) = \operatorname{Span}\left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right\}$

Both 1 dimensional kernels and observe that $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^2$.

Hence A is diagonalizable and for

$Q = \begin{bmatrix} 1 & 2/3 \\ -1 & 1 \end{bmatrix}$, $Q^{-1} =$

we have $Q^{-1} A Q = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

5.) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$

$\det(A - tI)$

$$= \det \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} -t & -2 & -3 \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{pmatrix} \end{array}$$

$$= (-t) \det \begin{pmatrix} 1-t & -1 \\ 2 & 5-t \end{pmatrix} - (-2) \det \begin{pmatrix} -1 & -1 \\ 2 & 5-t \end{pmatrix}$$

$$+ (-3) \det \begin{pmatrix} -1 & 1-t \\ 2 & 2 \end{pmatrix}$$

$= (-t)\left[(1-t)(5-t) + 2\right] + 2(t - 5 + 2) - 3\left((-1)(2) - (2)(1-t)\right)$

$= (-t)\left[5 - t - 5t + t^2 + 2\right] + 2(t-3) - 3(-2-2+2t)$

$= (-t)\left[7 - 6t + t^2\right] + 2t - 6 - 3(-4+2t)$

$= \cancel{-7t} + 6t^2 - t^3 \cancel{+2t} - 6 + 12 \cancel{-6t}$

$= -t^3 + 2t^2 - 11t + 6$

$t^3 - 6t^2 + 11t + 6 = 0 \quad \Rightarrow \quad (t-1)(t-2)(t-3) = 0.$

Hence eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$.

$\ker(A - I)$

$$= \ker\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} = \operatorname{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$\ker(A - 2I)$

$$= \ker\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$\ker(A - 3I)$

$$= \ker\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

where $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ forms a basis for $\mathbb{R}^3$.

Observe that for $Q := \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

$$Q^{-1} A Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Question 12 (§5.1, nr 5.)

Show that all these linear maps are diagonalizable

For each linear map $T: V \to V$ find the eigenvalues of T, and an ordered basis β for V s.t. $[T]_\beta$ is diagonal.

a.) $V = \mathbb{R}^2$, $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2a+b \\ -10a+9b \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -10 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

$$f_T(t) = \det\begin{pmatrix} t+2 & -1 \\ 10 & t-9 \end{pmatrix}$$

$$= (t+2)(t-9) + 10 \qquad \begin{bmatrix} -2 & 1 \\ -10 & 9 \end{bmatrix}$$

$$= t^2 - 7t - 18 + 10$$

$$= t^2 - 7t - 8$$

$$= (t-8)(t+1)$$

$\Rightarrow \lambda_1 = 8, \quad \lambda_2 = -1$

$$\ker\left(\begin{bmatrix} -2 & 1 \\ -10 & 9 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \ker\begin{pmatrix} -10 & 1 \\ -10 & 1 \end{pmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 10 \end{bmatrix}\right\}$$

$\ker\left(\begin{bmatrix} -2 & 1 \\ -10 & 9 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$= \ker\begin{pmatrix} -1 & 1 \\ -10 & 10 \end{pmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Then let $\beta = \left\{\begin{bmatrix} 1 \\ 10 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$,

$Q := [I]_\beta^\delta = \begin{bmatrix} 1 & 1 \\ 10 & 1 \end{bmatrix}$, where $\delta = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

and $[T]_\beta = Q^{-1} [T]_\delta^\delta Q$

$= \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}.$

j) $V = M_2(\mathbb{R})$, $T(A) = A^t + 2\,tr(A) I_2$.

First find matrix of T in standard basis:

$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \cdot 1 \cdot I_2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 2 \cdot 1 \cdot I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

So $[T]_{\mathcal{S}} = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

One can check that T has eigen values $4, -1, -1, 1$ and basis

$\left\{ \begin{bmatrix} 2/5 \\ 0 \\ 0 \\ 1/5 \end{bmatrix}, \begin{bmatrix} -1/5 \\ 0 \\ 0 \\ 2/5 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} \right\}$

Set $\quad Q := \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix}$

to get

$$Q^{-1}[T]_{\mathcal{S}}^{\mathcal{S}}Q = \begin{bmatrix} 4 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$$

Question 13 (S5.1, nr 9.)

Complete

Question 14 (§5.1 , 18)

Complete.

Exercise (§ 5.2, nr 20)

"The value of char poly f_A at 0 is det(A)".

Let A be a nxn matrix with characteristic polynomial

$$f_A(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$$

Prove that $f(0) = a_0 = \det(A)$

Hence A is invertable iff 0 is not a root of its characteristic polynomial.

(Also since roots = eigen vals, so say 0 is an eigenvalue if $a_0 = 0$.)

Proof: Recall that the characteristic polynomial

$$f_A(t) = \det(A - tI)$$

$$= (-1)^n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$$

So $f_A(0) = \det(A)$ and $f_A(0) = a_0$

Hence $\det(A) = a_0$. $\blacksquare$

Question 15 $(\S 5.1, \text{nr } 21)$

"Characteristic Polynomials see the trace too"!

Fix $A \in M_n(F)$, and f_A denote char poly. Prove that

a.) $f(t) = (a_{11} - t) \ldots (a_{nn} - t) + q(t)$
where $\deg(q) \leq n-2$.

b.) $tr(A) = (-1)^{n-1} a_{n-1}$, when expand f as
$f_A(t) = (-1)^n t^n + a_{n-1} t^{n} + \ldots + a_1 t + a_0$