Homework 12

Due April 18, 2025

Solutions and Rubric

Section 6.2 : 1., 2a., 6., 8., 19a.

Section 6.3 : 1, 2a., 3., 6., 7., 10, 15.

Grading Scheme

Question 6 (§6.3, nr. 1) Graded

a.) True, provided the underlying space is a inner product space

b.) False, ony when the codomain is 1 dimensional.

c.) False, we need the ordered basis β to be orthonormal, to see the connection between adjoints and conjugate transposes.

d.) True

e.) False, the adjoint $*: \mathcal{L}(W) \to \mathcal{L}(V)$ is conjugate linear.

F.) True, since the standard basis is o.n.

g.) True.

Question 7 (§6.3, nr. 2)

Complete

Question 8 (§ 6.3, nr. 3)

Graded

For each of the following inner product spaces V and linear maps $T : V \to V$, evaluate T^* at the given vector:

a.) $V = \mathbb{R}^2$, $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+b \\ a-3b \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, $x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, find T^*x.

Taking inner products allow us to compute T!

First entry

- $\langle e_1, T^*x \rangle = \langle Te_1, x \rangle = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\rangle$
$= 11$

second entry

- $\langle e_2, T^*x \rangle = \langle Te_2, x \rangle = \left\langle \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\rangle$
$= -12$

Hence $T^*x = \begin{bmatrix} 11 \\ -12 \end{bmatrix}$

b.) $V = \mathbb{C}^2$, $T\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2z_1 + i z_2 \\ (1-i) z_1 \end{bmatrix}$, $x = \begin{bmatrix} 3-i \\ 1+2i \end{bmatrix}$

$$= \begin{bmatrix} 2 & i \\ 1-i & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Compute

$$\langle e_1, T^* x \rangle = \overline{\langle T e_1, x \rangle}$$

working over $\mathbb{C}$ not $\mathbb{R}$

$$= \overline{\left\langle \begin{bmatrix} 2 \\ 1-i \end{bmatrix}, \begin{bmatrix} 3-i \\ 1+2i \end{bmatrix} \right\rangle}$$

$$= \left\langle \begin{bmatrix} 3-i \\ 1+2i \end{bmatrix}, \begin{bmatrix} 2 \\ 1-i \end{bmatrix} \right\rangle$$

$$= (3-i)\cdot \overline{2} + (1+2i)\cdot (\overline{1-i})$$

$$= 6 - 2i + (1+2i)(1+i)$$

$$= \cancel{6} - 2i + \cancel{1} + i + 2i + \cancel{2i^2}$$

$$= 5 + i$$

$$\langle e_2, T^* x \rangle = \overline{\langle T e_2, x \rangle}$$

$$= \overline{\left\langle \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 3-i \\ 1+2i \end{bmatrix} \right\rangle}$$

$$= \left\langle \begin{bmatrix} 3-i \\ 1+2i \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix} \right\rangle$$

$$= (3-i)\overline{i}$$

$= (3-i)(1-i)$

$= -3i - 1$

So $T^* x = \begin{bmatrix} 5+i \\ -3i-1 \end{bmatrix}$

In a, b, the standard basis is O.N., but not in $P_1(\mathbb{R})$.
↳ First need to orthogonalize.

c.) $V = P_1(\mathbb{R}) = \text{span}\{1, t\}$, $\langle f, g \rangle = \int_{-1}^{1} fg$, over $\mathbb{R}$.
$T(f) = f' + 3f$, $f(t) = 4 - 2t$.

We need an orthonormal basis to compute adjoints, using Gram-Schmidt we obtain
$\beta = \{ \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2} t \}$. Then

- $T(\frac{\sqrt{2}}{2}) = 3 \cdot \frac{\sqrt{2}}{2}$
- $T(\frac{\sqrt{6}}{2} t) = \frac{\sqrt{6}}{2} + 3 \frac{\sqrt{6}}{2} t$

Then $\langle \frac{\sqrt{2}}{2}, T^*(f) \rangle = \langle T(\frac{\sqrt{2}}{2}), f \rangle$

$= \langle 3 \cdot \frac{\sqrt{2}}{2}, 4 - 2t \rangle$

$= \int_{-1}^{1} 3 \cdot \frac{\sqrt{2}}{2} (4 - 2t)\, dt$

$= 12\sqrt{2}$ (coeff. next to

basis element $\frac{\sqrt{6}}{2}$)

And $\langle \frac{\sqrt{6}}{2}t, T^*(f)\rangle = \langle T(\frac{\sqrt{6}}{2}t), f\rangle$

$$= \langle \frac{\sqrt{6}}{2} + 3\frac{\sqrt{6}}{2}t, 4-2t\rangle$$

$$= \int_{-1}^{1}\left(\frac{\sqrt{6}}{2} + 3\frac{\sqrt{6}}{2}t\right)(4-2t)\,dt$$

$$= 2\sqrt{6}$$

Then $T^*(4-2t) = 12\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) + (2\sqrt{6})\frac{\sqrt{6}}{2}t$

$$= 12 + 6t.$$

Question 9 (§ 6.3, number 6.)

Graded

Turning non-selfadjoint elements into something that is self-adjoint.

Given a linear operator $T: V \to V$ on a inner product space V. Define two new operators
- $U_1 = T + T^*$
- $U_2 = TT^*$

and show that U_1 and U_2 are self-adjoint.

Proof: This follows from the basic properties of the adjoint operator on $\mathcal{L}(V)$.
Indeed

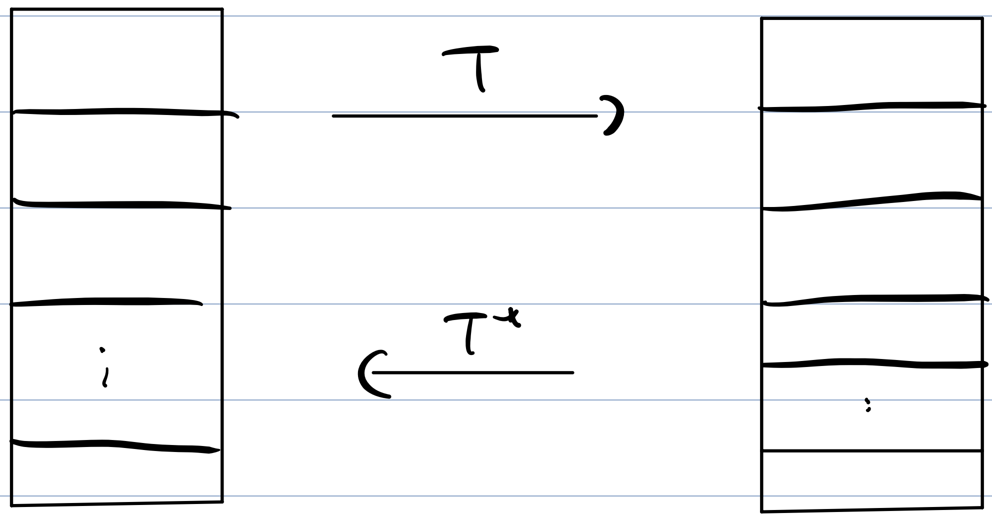
$$U_1^* = (T+T^*)^* \overset{\text{linearity}}{=} T^* + T^{**}$$
$$= T^* + T$$
$$= T + T^*$$
$$= U_1$$

and

$$U_2 = (TT^*)^* = (T^*)^* T^*$$
$$= TT^*$$
$$= U_2$$

Extra (§ 6.3, number 7.)

Give an example of a linear operator T on a finite dimensional inner product space V s.t. $\text{Ker}(T) \neq \text{Ker}(T^*)$



V V

Consider $T: \mathbb{R}^3 \to \mathbb{R}^3$, $\beta = \{e_1, e_2, e_3\}$ o.n. basis

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then $\text{Ker}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$[T]_\beta^* = [T^*]_\beta$ so

$$[T^*]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

when $\ker([T^*]_\beta) = \ker(T^*)$

$$= \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\neq \ker(T)$$

Question 10 (§ 6.3, nr. 8)

Adjoint of invertible operator is just the adjoint of its inverse.

Given a linear operator T on a finite dimensional inner product space V. If T is invertible

then the inverse of its adjoint T^* is $(T^{-1})^*$.

Observe that $(TT^{-1}) = I$

$$\Rightarrow \quad (TT^{-1})^* = I^* = I$$

$$\Rightarrow \quad (T^{-1})^* T^* = I$$

and $(T^{-1}T) = I \Rightarrow T^*(T^{-1})^* = I$.

Hence $(T^*)^{-1} = (T^{-1})^*$.

Remark: For complex numbers $z \neq 0$, we get

$$(\overline{z})^{-1} = \frac{1}{\overline{z}} = \overline{\left(\frac{1}{z}\right)} = \overline{(z^{-1})}.$$

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Question 11 (§6.3, nr 10)

Given a linear map $T: V \to V$ on a inner product space V. Show that

$\|T(x)\| = \|x\|$ for all $x \in V$ iff

$\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.

<u>Proof:</u> The converse follows immediately, since choose $y = x$, then $\forall x$,

$\langle T(x), T(x) \rangle = \langle x, x \rangle$

$\Rightarrow \|T(x)\|^2 = \|x\|^2$

$\Rightarrow \|T(x)\| = \|x\|.$

Now suppose that $\|T(x)\| = \|x\|$, $\forall x \in V$.

The first step is to show that for any linear $S: V \to V$, s.t. $S = S^*$ (self-adjoint) we get $\langle Sx, x \rangle = 0$ for all $x \in V$ implies $S = 0$.

Indeed let $x, y \in V$, and observe that

$$\langle S(x+y), (x+y) \rangle = 0$$

$$\Rightarrow \langle \cancel{Sx, x} \rangle + \langle Sy, x \rangle + \langle Sx, y \rangle + \langle \cancel{Sy, y} \rangle = 0$$

$$\Rightarrow \langle Sy, x \rangle + \langle Sx, y \rangle = 0$$

Since $S = S^*$,

$$\Rightarrow \langle y, Sx \rangle + \langle Sx, y \rangle = 0$$

$$\Rightarrow \overline{\langle Sx, y \rangle} + \langle Sx, y \rangle = 0$$

$$\Rightarrow 2\,\mathrm{Re}\langle Sx, y \rangle = 0, \qquad \mathrm{Re}(z) = \frac{z+\bar{z}}{2}$$

Also $\langle S(ix+y), ix+y \rangle = 0$

$$\Rightarrow \langle \cancel{S(ix), ix} \rangle + \langle S(ix), y \rangle + \langle Sy, ix \rangle + \langle \cancel{Sy, y} \rangle = 0$$

$$\mathrm{Im}(z) = \frac{z-\bar{z}}{2i} = -i\left(\frac{z-\bar{z}}{2}\right)$$

$$\Rightarrow \langle S(ix), y \rangle + \langle Sy, ix \rangle = 0$$

$(S = S^*)$

$$\Rightarrow \langle S(ix), y \rangle + \langle y, S(ix) \rangle = 0$$

$$\Rightarrow \langle S(ix), y \rangle + \overline{\langle S(ix), y \rangle} = 0$$

$$\Rightarrow i\langle S(x), y \rangle + \bar{i}\,\overline{\langle S(x), y \rangle} = 0$$

$$\Rightarrow i\langle S(x), y \rangle - i\,\overline{\langle S(x), y \rangle} = 0$$

$\Rightarrow i\left(\langle S(x),y\rangle - \overline{\langle S(x),y\rangle}\right) = 0$

$\Rightarrow 2\,\mathrm{Im}\left(\langle S(x),y\rangle\right) = 0$

Since $\mathrm{Re}(\langle Sx,y\rangle) = 0$ and $\mathrm{Im}(\langle Sx,y\rangle) = 0$ we get $\langle Sx,y\rangle = 0$. Since this holds for all $x,y \in V$ this means $S = 0$.

Now if we assume $\|Tx\| = \|x\|$ $\forall x \in V$, observe that

$\langle T(x),T(x)\rangle = \langle x,x\rangle \quad , \forall x$

$\Rightarrow \langle (T^*T - I)x, x\rangle = 0 \quad , \forall x$

$\Rightarrow$ By above since $(T^*T-I)^* = T^*T-I$, we get $T^*T = I$.

Then $\langle T(x), T(y)\rangle$

$= \langle x, T^*Ty\rangle$

$= \langle x, y \rangle \qquad , \forall x, y \in V.$

$\boxed{S}$

In Thm 6.9, the linear map $T: V \to V$ had the same domain and codomain. This next example shows that this is not necessary to define adjoints.

Question 12 (§ 6.3, nr 15)

Completion

Given a linear transformation $T: V \to W$ between two inner product spaces V an W. A linear $T^*: W \to V$ is called an adjoint of T whenever

$$\langle Tx, y\rangle_W = \langle x, T^*y\rangle_V \quad \text{for all}$$

$x \in V$, $y \in W$.

a) Show that T has a unique adjoint.

b.) If β and γ are two o.n. bases for V and W, respectively, then

$$[T^*]_\gamma^\beta = \left([T]_\beta^\gamma\right)^*$$

c.) $\mathrm{Rank}(T) = \mathrm{Rank}(T^*)$

d) $\langle T^*x, y\rangle_V = \langle x, Ty\rangle_W$, $\forall x \in W, y \in V$.

e) For all $v \in V$, $T^*T(v) = 0$ iff $T(v) = 0$

Proof: Fix $w \in W$, and define a linear functional

$$v \mapsto \langle Tv, w \rangle_W : V \to \mathbb{C}$$

Since this is a linear functional on V, by Riesz representation $\exists!$ vector in V call it $T^*w \in V$ s.t

$$\langle v, T^*w \rangle = \langle Tv, w \rangle \quad , \forall v \in W.$$

Define $T^* : W \to V$ to be the map that send $w \to T^*w$.

One needs to show T is linear:

- Fix vectors $w_1, w_2 \in W$. The vector $T^*(w_1 + w_2)$ is the only vector with the property that

$$\langle v, T^*(w_1 + w_2) \rangle = \langle Tv, w_1 + w_2 \rangle, \quad \forall v \in V. \quad (*)$$

- Futhermore vectors $T^*(w_1)$, $T^*(w_2)$ has the property that
  - $\langle v, T^*(w_1) \rangle = \langle Tv, w_1 \rangle \quad , \forall v \in V$
  - $\langle v, T^*(w_2) \rangle = \langle Tv, w_2 \rangle \quad , \forall v \in V$ $\Bigg\} (*)$

- By conjugate linearity in the second component

$$\langle v, T^*(w_1)+T^*(w_1)\rangle = \langle v, T^*(w_1)\rangle + \langle v, T^*(w_2)\rangle$$
$$\overset{*}{=} \langle Tv, w_1\rangle + \langle Tv, w_2\rangle$$
$$= \langle Tv, w_1+w_2\rangle, \quad \forall v \in V$$

showing $T^*(w_1)+T^*(w_1)$ satisfy the property $*$, and since $T^*(w_1+w_2)$ is the only vector that does, we conclude $T^*(w_1)+T^*(w_1) = T^*(w_1+w_2)$
(Similar idea for scalar multiplication)

- For uniqueness of T^*, it follows from the fact that $\langle Sx, y\rangle = 0$ for all $x, y \in V$ implies $S=0$.

b.) Let $\beta = \{v_1, v_2, \dots, v_n\}$ o.n. basis for V, and $\gamma = \{w_1, w_2, \dots, w_m\}$ o.n. basis for W.

- We show that the (i,j) entry of $[T]_\beta^\gamma$ is $\langle Tv_i, w_j \rangle$ and the (i,j) entry of $[T^*]_\gamma^\beta$ is $\langle T^* w_i, v_j \rangle$

Indeed because $T^* w_i = \sum_{k=1}^{n} \alpha_k v_k$, unique expansion of $T^* w_i \in V$.

Recall that the i^{th} column of $[T^*]_\gamma^\beta$ the co-ordinate vector

$$[T^* w_i] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}$$

So α_j is the (i,j) entry of $[T^*]_\gamma^\beta$.

- We can find α_j by taking the inner-product with v_j! Indeed by orthonormality

$$\langle T^* w_i, v_j \rangle = \sum_{k=1}^{n} \alpha_k \langle v_k, v_j \rangle = \alpha_k$$

since $\langle v_k, v_j \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$

In general the $(i,j)^{th}$ entry of $[T^*]_\gamma^\beta$ is

$$\langle T^* w_i, v_j \rangle$$

A similar argument shows that the $(i,j)^{th}$ entry of $[T]_\beta^\gamma$ is

$$\langle T v_i, w_j \rangle.$$

Here we can conclude $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$

since the (i,j) entry of $([T]_\beta^\gamma)^*$

is $\overline{\langle T v_j, w_i \rangle}$ (entry when conj. transpose)

$= \overline{\langle v_j, T^* w_i \rangle}$

$= \langle T^* w_i, v_j \rangle$

which is the (i,j) entry of $[T^*]_\gamma^\beta$!

$\blacksquare$

c.) $\mathrm{Rank}(T^*) = \mathrm{Rank}(T)$ follows from the calculation above, and the fact that conjugate transposing a matrix does not change its Rank.

Indeed $\mathrm{Rank}(T^*) = \mathrm{Rank}([T^*]_\gamma^\beta)$ part b.

$= \mathrm{Rank}(([T]_\beta^\gamma)^*)$ conj. transp does not change rank

$= \mathrm{Rank}(([T]_\beta^\gamma))$

$= \mathrm{Rank}(T)$.

d.) We know

$$\langle Ty, x\rangle_W = \langle y, T^* x\rangle_V \quad , \forall\ y \in V, x \in W$$

$$\Rightarrow \overline{\langle Ty, x\rangle_W} = \overline{\langle y, T^* x\rangle_V} \quad , \text{taking complex conj}$$

anti symmetric

$$\Rightarrow \langle x, Ty\rangle_W = \langle T^* x, y\rangle \quad , \text{as required.}$$

e.) Firstly, if $T(x)=0$, since T^* is linear the result follows. Conversely suppose $T^*T(x)=0$.

Recall the the only vector orthogonal to every vector is the zero vector. Lets show that our assumption implies $T(x) \perp W$

We decompose $W = R(T) \oplus R(T)^{\perp}$.
For $y \in R(T)^{\perp}$ then

$$\langle T(x), y \rangle = 0 \quad \text{since} \quad T(x) \in R(T).$$

For $y \in R(T)$ we get $y = T(z)$ some $z \in V$.
Then

$$\langle T(x), y \rangle =$$

As required to show $T(x) \perp W$ which implies $T(x)=0$.

Extra: (§ 6.3, nr 12.)

Given a linear operator $T: V \to V$ on an inner product space V. Then

1.) $R(T^*)^\perp = N(T)$

2.) If V is finite dimensional, then
$R(T^*) = N(T)^\perp$ (For $\dim(V) = \infty$, $\overline{R(T^*)} \overset{(1.)}{=} N(T)^\perp$)

Proof: