Homework 1

Jan 24, 2025

Rubric

1.2 : 1, 7, 11, 12, 13, 18

1.3 : 1, 5, 8, 11, 12, 13, 15, 17, 22

Grading Scheme

- Hw counts 8 marks on Canvas
- 5 points per problem
- Total 50 points
- So final score is $\left(\frac{\text{score}}{50}\right) \cdot 8$

§1.2	1.	Graded	15.	Not graded
	7.	Graded	17.	Graded
	11.	Completion	22.	Not graded
	12.	Completion		
	13.	Graded		
	18.	Not graded		
§1.3.	1.	Graded		
	5.	Not graded		
	8.	Completion		
	11.	Graded		
	12.	Not graded		
	13.	Graded		

Question 1 (§1.2 , nr. 1)

a.) True, by definition (VS3).

b.) False, by Cor 1, page 12.

c.) False, take the vector $x = 0 \in V$.

d.) False, take the scalar $a = 0 \in F$.

e.) True.

f.) False, rows by columns.

g.) False, $(3x^2 + x + 1) + (x+1) = (3+0)x^2 + (1+1)x + 1+1$.

h.) False, $(-3x^2+1) + (3x^2 + x) = x+1$, degree < 3.

i) True.

j) True.

k) True.

Start at 5 points and lose -0.5 for each incorrect.

Rubric Question 1
- Start with 5 points, and -0.5 for each incorrect.

Question 2 (§1.2, nr 7.)

$\mathcal{F}(S, \mathbb{R})$ – All real valued functions from $\{0,1\} \to \mathbb{R}$.

- $f(t) := 2t+1$
- $g(t) := 1+4t-2t^2$
- $h(t) := 5^t + 1$

Equality and Algebraic operations are defined pointwise.

How do we define equality? We say $f = g$ in $\mathcal{F}(S, \mathbb{R})$ to mean $f(t) = g(t)$ $\forall$ $t \in S$.

Proof:

To show $f=g$ in $\mathcal{F}(S, \mathbb{R})$ we need to show $f(t) = g(t)$ for all $t \in S$. Indeed

$f(0) = 1$ and $g(0) = 1$ $\Rightarrow$ $f(0) = g(0)$

$f(1) = 3$ and $g(1) = 3$ $\Rightarrow$ $f(1) = g(1)$

So $f(t) = g(t)$ $\forall$ $t \in S = \{0,1\}$.

And hence $f=g$ in $\mathcal{F}(S, \mathbb{R})$.

Similarly check:

$(f+g)(0) = 2 = h(0)$

$(f+g)(1) = 6 = h(1)$.

$\blacksquare$

Rubric Question 2

- 1 point, completion
- 1 point each
  - $f(0) = g(0)$
  - $f(1) = g(1)$
  - $(f+g)(0) = h(0)$
  - $(f+g)(1) = h(1)$

Question 3 (§1.2, nr. 11)

Let $V = \{0\}$ consisting out of a single vector and define $0+0=0$ and $c \cdot 0 = 0$, $\forall c \in F$.

Prove that V is a vectorspace over F.

Proof:

0.) Check that $+ : V \times V \to V$ and $\cdot : F \times V \to V$ well def.

(VS 1) Associty :

$(0+0)+0 = 0+0 = 0 \Rightarrow (0+0)+0 = 0+(0+0)$

and $0+(0+0) = 0+0 = 0$.

(VS 2) Commuting: $0_1 + 0_2 = 0 = 0_2 + 0_1$

(VS 3) 0 is the addit identity.

(VS 4) The addit inverse of $0 \in V$ is 0,

(VS 5) $(ab)x = a(bx)$?

$(ab)0 = 0$ and $a(b0) = a \cdot 0 = 0$

(VS 6) $a(x+y) = ax + ay$ well $a(0+0) = a \cdot 0 = 0$ and $a \cdot 0 + a \cdot 0 = 0 + 0 = 0$

(VS 7) $(a+b) \cdot 0 = 0 = a \cdot 0 + b \cdot 0$

(VS 8) $1 \cdot 0 = 0$

$\boxtimes$

Question 4 (§1.2, nr 12)

let $E := \{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f(-t) = f(t) \ \forall t \in \mathbb{R} \}$

Show that E is a vector space under pointwise addition and scalar multiplication.

Proof:

let $z : \mathbb{R} \to \mathbb{R}$ by $z(t) = 0 \quad \forall t \in \mathbb{R}$.

Then for any $f \in E$ we have $z + f$ defined by $(z+f)(t) := z(t) + f(t)$. But then $(z+f)(t) = 0 + f(t)$, $\forall t$

$\Rightarrow z + f = f$ in E.

Also $z(-t) = 0 = z(t) \quad \forall t \Rightarrow z \in E$.

- Distribution, associativity follow from $\mathbb{R}$.
- Check $(f+g)(-t) = f(-t) + g(-t) \underset{f, g \in E}{=} f(t) + g(t) = (f+g)(t)$

So $f + g \in E$.

- Same with cf.

By subspace test, E subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

$\boxtimes$

Question 5 (§1.2, nr 13)

No, we can find an axiom in the definition of vector space that V does not satisfy:

Proof: We show that V does not have an additive identity (VS 3).

First observe that

$$(1,0)+(0,1) = (1+0,\ 0\cdot 1) = (1,0).$$

and $$(0,1)+(1,0) = (0+1,\ 1\cdot 0) = (1,0)$$

showing that $(0,1)$ is a candidate for the additive identity. By uniqueness its the only candidate.

But $(0,1)$ is not the additive identy, because otherwise $(1,0)+(a,b)=(0,1)$ for some $(a,b)\in V$. Which is a contradiction since $0\cdot b \neq 1$, for any $b\in\mathbb{R}$.

$\boxtimes$

Method 2: (Fails a distributive law)

Proof: Recall VS-8 where

$$(\alpha+\beta)v = \alpha v + \beta v, \quad \text{for all scalars } \alpha,$$

and vectors v.

$(a+b)(x,y) = a(x,y) + b(x,y)$

$(\alpha+\beta)(x,y)$ $\quad$ $(ax,y)+(bx,y)$

$= ((\alpha+\beta)x, y)$ $\quad$ $= (ax+bx,\ y\cdot y)$

Observe that $\alpha = \beta = 2$, $\quad v = (3,4)$

$$(2+2)(3,4) = ((2+2)3, 4) = (12, 4)$$

but

$$\begin{aligned}
2\cdot(3,4) + 2\cdot(3,4) &= (2\cdot 3, 4) + (2\cdot 3, 4) \\
&= (6,4) + (6,4) \\
&= (6+6,\ 4\cdot 4) \\
&= (12, 16) \\
&\neq (12, 4).
\end{aligned}$$

$\boxtimes$

Rubric

- 2 points for correcly determine vector space or not.
- 3 points for justification.

* class notes:

- To prove something is not a "____", you need to provide a concrete example!

Question 6 (§1.2, nr 18)

Given a set $V = \{(a_1, a_2) \mid a_1, a_2 \in F\}$

for some field F.

Define

$$(a_1, a_2) + (b_1, b_2) := (a_1 + 2b_1, a_2 + 3b_2)$$

and

$$c(a_1, d_2) := (ca_1, ca_2)$$

Is V a vector space over $\mathbb{R}$ with these operations?

No, $+ : V \times V \to V$ is not commutative.

Take $F = \mathbb{R}$,

$$(1,1) + (2,2) = (1 + 2(2), 1 + 3(2))$$
$$= (5, 7)$$

$$(2,2) + (1,1) = (2 + 2(1), 2 + 3(1))$$
$$= (4, 5)$$
$$\neq (5, 7)$$

Showing that V is not a vector space.

Question 7 (§1.3, nr. 1)

a.) False, W needs to be over same field, same operations as V

b.) False, no additive identity.

c.) True, choose $W = \{0\} \neq V$

d.) False, $[-1,1] \cap [0,2] = [0,1]$ not a subs of $\mathbb{R}$.

e.) True, $[A]_{1,1}, [A]_{2,2}, \dots, [A]_{n,n}$ are only possible $\neq 0$ entries

f.) False,

$Tr(A) = \sum_{i=1}^{n} [A]_{i,i}$, $\forall A \in M_{n \times n}(F)$

so for $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 \neq 1 = 1 \cdot 1$

g.) False, as sets $W \subseteq \mathbb{R}^3$ so $W \neq \mathbb{R}^2$

Question 8 (§1.3, nr 5.)

Show that $A + A^t$ is symmetric for all square matrices A.
Recall that symmetric means $B^t = B$.

- By ex 3, $(aA + bB)^t = aA^t + bB^t$.
  - Indeed summing first and then transposing is the same as transposing, then summing
- Also $(A^t)^t = A$.

Thus

$$(A + A^t)^t = A^t + (A^t)^t = (A^t)^t + A^t$$

and

$$(A + A^t)^t = A + A^t$$, as required.

As required.

Completion

Question a (§1.3, nr. 8)

a.) $W_1 = \left\{ \vec{a} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} a_1 = 3a_2 \\ a_3 = -a_2 \end{matrix} \right\}$

$= \left\{ \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

Yes, its a line through the origin.

b.) $W_2 = \left\{ \begin{bmatrix} t+2 \\ s \\ t \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$

No, $0 \notin W_2$

c.) $W_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix} = 0 \right\}$

Yes, W_3 forms a plane in $\mathbb{R}^3$.

Chose $h = \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix}$, and define $h^* : \mathbb{R}^3 \to \mathbb{R}$ by

$h^*(v) = \langle v, h \rangle$, so $W_3 = \ker(h^*)$

d.) Yes, W_4 is a subspace, same idea as c.)

e) No, does not contain 0 since

$0 + 2(0) - 3(0) = 0 \neq 1$

$W_5 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; a_1 + 2a_2 - 3a_1 = 1 \right\}$ is called an "affine plane".

f.) No, not closed under scalar multiplication.

$$W_6 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \left\langle \begin{bmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} \right\rangle = 0 \right.$$

well $5a_1^2 - 3a_2^2 + 6a_1^2$

$a_1 = 1$

$a_2 = 2$

$$5(1)^2 - 3(2)^2 + 6(a_3)^2 = 0$$

$$5 \quad -12 \quad + 6a_1^2 = 0$$

$$6a_3^2 = 7$$

$$a_3 = \sqrt{\frac{7}{6}}$$

So $\begin{bmatrix} 1 \\ 2 \\ \sqrt{\frac{7}{6}} \end{bmatrix} \in W_6$ $\quad \begin{bmatrix} 2 \\ 4 \\ 2\sqrt{\frac{7}{6}} \end{bmatrix} \notin W_6$ since

30

+18

$$5(2)^2 - 3(4)^2 + 6\left(2\sqrt{\frac{7}{6}}\right)^2$$

$$= 10 - 3 \cdot 16 + \cancel{6} \cdot 4 \cdot \frac{7}{\cancel{6}}$$

$$= 10 - 48 + 28$$

$$= 10 - 20$$

$$= -10 \neq 0$$

$\Rightarrow$ not closed under scalar multipl.

Question 10 (§ 1.3, nr 11.)

Recall $P(F) := \{$ polynomials with coeff in $F\}$ where F is a field.

Fix $n \geq 1$.

Is $W_n := \{ f \in P(F) \mid f = 0$ or $\deg(f) = n \}$

Well W_n has the zero polynomial.

But $(x^n + x^{n-1}) + ((-1)x^n + x^{n-1}) = x^{n-1} \neq 0$

nor have degree n. Hence W_n not closed under addition.

Rubric

- 2 points for stating not a vector space
- 3 points for justification.

Question 11 (§1.3, nr 12.)

Prove that the set of $m \times n$ upper triangular matrices is a subspace of $M_{m \times n}(F)$.

① The zero matrix is upper triangular.
② Summing two triangular matrix leaves it triangular, since $0+0=0$ in all entries needed to be 0.
③ Scalar multpl leave 0's in correct entr.
$\Rightarrow$ By subspace test, we have a subsp.

Question 12 ($\S$ 1.3, Nr. 13)

Hint: Apply the subspace test.

Fix $s_0 \in S$ and

let $W := \{ f \in \mathcal{F}(S, F) \mid f(s_0) = 0 \}$.

W is a subset of the vector space $\mathcal{F}(S,F)$

- The zero function in $\mathcal{F}(S,F)$ is the function $f : S \to F$ defined by $f(s) = 0$ $\forall s \in S$. In particular $f(s_0) = 0 \Rightarrow W$ contains the zero function.

- Let $g, h \in W$. Then

  $(g+h)(s_0) = g(s_0) + h(s_0) = 0 + 0 = 0$

  Hence $g + h \in W$.

- Let $\alpha \in F$. Then $(\alpha g)(s_0) = \alpha(g(s_0)) = \alpha \cdot 0 = 0$

  Thus $\alpha g \in W$.

By subspace test, W is a subspace of V.

Rubric

- 2 points conclusion
- 3 points justification.

Question 13 (§1.3, nr. 15)

Not Graded

Proof: Let $W := \{ f : \mathbb{R} \to \mathbb{R} \mid f' \text{ exists} \}$

The following follows from Calculus I:

- Sine differentiable functions are continuous we know $W \subseteq \mathbb{R}$.
- The zero function is differentiable since all constant functions are differentiable.
- Recall that derivatives are defined through limits. And by limit laws, it follows that the sum and scala multiplication of differentiable functions are differentiable.

- Recall that for $f \in W$, we define the derivative:

$$f'(t) := \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

So if $g \in W$, then

$$(f+g)'(t) = \lim_{h \to 0} \frac{(f+g)(t+h) - (f+g)(t)}{h}$$

$$= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} + \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= f'(t) + g'(t)$$

Not needed for hw submission.

By the subspace test, it follows that W is a subspace of $C(\mathbb{R})$. $\boxtimes$

Question 14 (§1.3, nr. 17)

The goal of this exercise is to see that the condition $W \neq \emptyset$ can be used instead of $0 \in W$.

Proof: Let V be a vector space over a field F and $W \subseteq V$.

If W is a subspace, the three assumptions follows from definition.

For the converse, suppose $W \neq \emptyset$, and that $x+y,\ ax \in W$ for all $x, y \in W$, $a \in F$.

Since $W \neq \emptyset$, we know $\exists\ v \in W$.

Consider the additive inverse in V of v, denoted $-v$.

Since $-v = (-1)v$ we know the additive inverse of v is in W. Then $v + (-v) = 0 \in W$.

By the subspace test we know W is a subspace of V $\boxtimes$

Rubric

- 1 point W subspace $\Rightarrow$ 3 conditions
- 2 points using W closed under addition
- 2 points using W closed under scalar mult.

or 4 points for the following:

- Assume $W \subseteq V$, V vector space, $W \neq \phi$, $ax, x+y \in W$ $\forall x, y \in W$, $a \in F$.
- Take $x \in W$. Then $0 \cdot x \in W \subseteq V$. So $0 \cdot x \in V \overset{\text{Thm}}{\underset{1.2}{\Rightarrow}} 0x = 0 \Rightarrow 0 \in W$.

Must justify why $0 \cdot x = 0$.

Question 15 (§1.3, nr. 22)

Not Graded

Proof: let $E := \{ f : F_1 \to F_2 \mid f \text{ is even} \}$

and $\mathcal{O} := \{ f : F_1 \to F_2 \mid f \text{ is odd} \}$.

Recall that $z \in \mathcal{F}(F_1, F_2)$ is the function s.t. $z(t) = 0$ for all $t \in F_1$.

So $z(-t) = 0 = z(t)$ and $z(-t) = 0 = -z(t)$ for all $t \in F_1$ implies that $z \in E$ and $z \in \mathcal{O}$.

Next let $f, g \in E$. Observe that for $t \in F_1$,

$$
\begin{aligned}
(f+g)(-t) &= f(-t) + g(-t) && \text{, def of + in } \mathcal{F} \\
&= f(t) + g(t) && \text{since } f, g \in E \\
&= (f+g)(t)
\end{aligned}
$$

Thus $f+g \in E$. Also for $\alpha \in F_2$, we have

$$
\begin{aligned}
(\alpha f)(-t) &= \alpha(f(-t)) \\
&= \alpha(f(t)) \\
&= (\alpha f)(t)
\end{aligned}
$$

which shows $\alpha f \in E$. By the subspace test it follows that E is a subspace of $\mathcal{F}(F_1, F_2)$.

A similar argument shows $\mathcal{O}$ is a subspace of $\mathcal{F}(F_1, F_2)$

$\boxtimes$